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ALGEBRAIC FUNCTIONS OF H-FUNCTIONS WITH

SPECIFIC DEPENDENCY STRUCTURES

A practical technique is presented for determining the
probability density function and cumulative distribution function of
products, quotients, and powers of dependent random variables with
bivariate H-function distributions. The bivariate H-function is the
most general function of two variables, encompassing as special cases
most of the other special functions of mathematics and many of the
classical bivariate distributions. The unique properties of the
bivariate H-function make it a powerful tool in the analysis of
products, quotients, and powers of dependent random variables.

This dissertation first provides background material, including
history, on the algebra of random variables, definitions and
properties of double integral transforms, and theorems on
transformations of random variables. The history of bivariate·
H-functions along with two new definitions, associated properties, and
special cases of the bivariate H-functions are given. Theorems
expanding the use of double Mellin transforms to find the probability

density function of products, quotients, and powers of dependent random variables are given. Included in this work are bivariate transformation theorems for products and ratios of pairwise independent variables from two or more bivariate distributions.

The definition, special cases, and transformation theorems for the bivariate H-function distribution are presented. These theorems show that the probability density function of products, quotients, and powers of dependent H-function variates is an H-function distribution of one variable. Transformation theorems for products and ratios of pairwise independent H-function variates from two or more bivariate H-function distributions are also given. Such combinations of pairwise independent variables result in bivariate distributions which are also bivariate H-functions. Formulas for finding the ordered moments of the bivariate H-function distribution are derived. The cumulative distribution function of a bivariate H-function distribution is shown to be another bivariate H-function. The cumulative distribution function is then used to derive a formula for finding the constant of a bivariate H-function distribution.

Utilizing theorems from complex analysis of higher dimensions, the analytic form of the bivariate H-function is analyzed by performing the double contour integral iteratively. In this fashion inversion is accomplished by summing the residues iteratively in each of the complex s planes.

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ALGEBRAIC FUNCTIONS OF H-FUNCTIONS WITH

SPECIFIC DEPENDENCY STRUCTURES

Stuart Duane Kellogg, B.S., M.B.A., M.S.

(Captain, USAF, 213 pages)

DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN

May 1984

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ALGEBRAIC FUNCTIONS OF H-FUNCTIONS WITH

SPECIFIC DEPENDENCY STRUCTURES

APPROVED BY SUPERVISORY COMMITTEE

To God

for granting me the opportunity of life,
for providing the faith to sustain that life:

  my parents, Clifford and LaVonne,
    for their strength and wisdom
  my wife, Mary,
    for her love and support
  my children, Erin and Brendan,
    for their unbounded love
  my supervising professor, Wes Barnes,
    for his friendship.

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Publication No. _____

Stuart Duane Kellogg, Ph.D.
The University of Texas at Austin, 1984

Supervising Professor: J. Wesley Barnes

A practical technique is presented for determining the
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CHAPTER 1

Introduction and Review

1.1 Purpose and Scope

Suppose one wishes to determine the exact probability density
function of the product of two random variables, X and Y, with known
probability density functions, $f_X(x)$ and $f_Y(y)$, such that $f_X(x) = 0$
for $x < 0$ and $f_Y(y) = 0$ for $y < 0$. If X and Y are independent, the
desired answer is the inverse Mellin transform of the product of the
Mellin transforms of $f_X(x)$ and $f_Y(y)$. This process is well
established and has been used extensively in the algebra of
independent variables.

Now, suppose that X and Y are not independent. If the
bivariate probability density function $f_{X,Y}(x,y)$ is known, an answer
still may be obtained by the use of double Mellin transform
techniques. To date, very little work has been done in this area due
to the difficulties of performing the double transform operations.

Suppose, however, that one has a general function of two
variables which has as special cases all of the bivariate probability
density functions in some group of interest. If a solution exists for
the general function of two variables by application of the double
transform techniques, then the resulting solution covers all those
problems involving the special cases. This is the motivation for
using a general function of two variables.

Suppose further, that the application of the double transform

techniques to the general function of two variables has as its

solution a general function of one variable which has as special cases

all the univariate probability density functions of some interest

group; specifically, the H-function of one variable. If this

supposition is true, then one has a powerful technique for finding

exact distributions of products and quotients of dependent random

variables, and a means of combining these distributions with other

independent univariate distributions.

The primary purpose of this dissertation is to develop a

general technique, presented in Chapter 5, for determining the

probability density function and the cumulative density function of

the random variable

$$Z = X^p Y^q$$

where X and Y are dependent random variables with joint probability

density function which may be expressed as a bivariate H-function and

p and q are rational constants. The general function of two variables

known as the bivariate H-function is chosen for several reasons.

First, the bivariate H-function is the most general of the special

functions of two variables and includes nearly every named function as

a special case. Second, much like the bivariate normal arises as a

natural extension to the univariate normal, it would seem that the

bivariate H-function distribution should be a natural extension to the

univariate H-function distribution. This is indeed the case. Chapter

4 shows that many of the properties that hold for univariate

H-function distributions hold also for the bivariate H-function distribution. The bivariate H-function distribution has the additional feature that the products, quotients and rational powers of dependent H-function variates are reduced to univariate H-function variates.

In developing the theory above, a second purpose evolved as a natural extension to the primary purpose above. A technique is presented in Chapter 5 for determining the bivariate probability density function for the random variables

$$Z = \prod_{i=1}^{n} X_i^{P_i} \quad , \quad W = \prod_{i=1}^{n} Y_i^{q_i}$$

where each pair, (X_i, Y_i), are dependent variates with a given bivariate density function which is expressible as a bivariate H-function and X_i, X_j $i \neq j$ and Y_i, Y_j $i \neq j$ are independent. Exponents P_i and q_i are rational constants. It is shown in Chapter 5 that such combinations of pairwise independent variates result in the dependent variates Z and W which have a bivariate density function which is also expressible as a bivariate H-function distribution.

In the course of developing the above general technique, some secondary purposes became evident. One is the attempt, in Chapter 3, to relate bivariate H-functions to known special functions and to other simpler H-functions. The general form for the bivariate H-function is a double contour integral containing products and

quotients of gamma functions and is not readily identified by this form. The properties and identities given in Chapter 3 prove useful in evaluating H-functions in later Chapters.

In his dissertation, Cook (5) gives a convergence proof for the univariate H-function distribution and a readily applied technique for inverting the H-function using residue theory. Chapter 6 demonstrates the applicability of using residue theory in the bivariate case and several examples are given. While the techniques given are generally applicable, a bivariate proof similar to that given by Cook would still be required to develop practical guidelines for when left half plane residues versus right half plane residues should be summed in each contour integral to evaluate a given bivariate H-function.

To study and develop the power of the bivariate H-function distribution, three new bivariate exponential distributions are developed in Appendix C. These distributions are shown to be special cases of the bivariate H-function distribution in Chapter 4. These distributions indicate the versatility of the bivariate H-function distribution in the contours the bivariate H-function can undertake as given in Appendix D.

Some important limitations to the scope of this dissertation must be stated. For instance, techniques are presented for determining products, quotients and rational powers of dependent H-function variates. The study of sums and differences of independent

H-function variates has been established through the work of Carter (3,4), Eldred (7), and Cook (5). The extension to dependent H-function variates for sums and differences is not immediately accomplished. To do so, one must first extend Prasad's theorems (57) for obtaining the Mellin transform from its Laplace transform and conversely to the bivariate case. Then, one must be able to obtain the Laplace transform of the bivariate H-function. Some work in this area has already been accomplished by Goyal (69). Finally, one must determine if a general technique for using double Laplace transforms exists for solving for the density function of a random variable which is the sum of two dependent random variables. While it would seem natural that such a theory similar to that developed for products and quotients of dependent random variables using double Mellin transforms exists, the development of such a theory would present a formidable task.

The bivariate H-function is not defined for a zero or negative real value of its arguments. Therefore, only probability density functions that are defined to be zero for nonpositive arguments are treated. Techniques for finding probability density functions defined non-zero for both positive and negative arguments are handled by dividing such functions into four components, one for each quadrant. Such techniques are presented by Fox (10) and Subrahamian (19). These techniques result as an extension to work done by Epstein (9) and Springer and Thompson (18) for independent variables.

The algebra of random variables is a vast field of study and the study of the algebra of _dependent_ random variables is still in a relative state of infancy. Combining the advantages of a general function and of certain properties of the bivariate H-function and its subsequent reduction to a univariate H-function for products and quotients of dependent H-function variates is, hopefully, a meaningful contribution to this field of study.

1.2 Literature Survey

Since the 1920's, considerable attention has been given to the derivation of probability distributions that are the result of some algebraic combination of random variables with known probability distributions. Early authors, including Aroian (24), Craig (26), and Craig (27,28,29) have presented detailed discussions on sums, products, and quotients of independent random variables. The use of Fourier and Laplace transforms extended the earlier work dealing with sums and differences of random variables. Springer provides an excellent discussion and bibliography on this subject in his book (17).

The problem of treating products and quotients of random variables, however, was limited to a few special cases. The first practical approach for dealing with products and quotients of independent variables was presented by Epstein in 1948 (9). Epstein used the Mellin integral transform to derive the probability density functions of the Student t and Fisher F statistics and of the product of two standardized normal variates. His work was limited to two random variables. In 1966, Springer and Thompson (18) extended the work of Epstein to n random variables.

The most significant advances in the algebra of random variables came in 1972 when Carter (3,4) tied together the physical science work on H-functions and the probability work on Mellin integral transforms into a powerful general theory. He introduced a

new probability distribution, the H-function distribution, which
includes as special cases, ten of the well known classical
distributions – gamma, exponential, chi-square, Weibull, Rayleigh,
Maxwell, half-normal, beta, half-Cauchy, and general hypergeometric.
Carter also proved that products, quotients, and rational powers of
independent H-function variates are also H-functions.

In 1979, Eldred (7) extended the work of Carter by developing
a computer program to calculate the probability density function of
combinations of products, quotients, and powers of H-function
variates. He also derived the H-functional form for the half-Student
and F distributions.

Cook (5,6) carried this work even further to provide a simpler
method for calculating the H-function resulting from some combination
of H-function variates. He also developed a new computer program
which could handle sums as well as products, quotients, and powers of
H-function variates.

Most of the current work on transform and H-function theory
for the algebra of independent random variables is given in books by
Springer (17), Mathai and Saxena (14), and Giffin (98) and in papers
by Eldred (7) and Cook (5,6).

Today, the H-function techniques are powerful enough to handle
most algebraic combinations of independent random variables. However,
the algebra of dependent random variables has received little
attention. Much of this has been due to the inability to separate

random variables residing in a multivariate density function due to the dependency structure of the density function. Indeed, simply defining a bivariate density function for two dependent random variables with known marginal density functions has been a major obstacle. Except for the bivariate normal, no unique bivariate density function can be derived for two random variables with a given covariance matrix and marginal density functions.

In the 1920's, other bivariate distributions were constructed which had as marginal distributions corresponding well known univariate distributions which included - bivariate Students t, bivariate beta, and Rhodes distributions. Little more was accomplished until the development of a bivariate gamma distribution in 1941.

In 1960, Gumbel (37) studied a bivariate distribution which has exponential margins, but no meaningful derivation for the distribution is known. Marshall and Olkin (48,49) introduced a bivariate distribution with exponential marginals by studying a two-component system which fails to function after a shock to one or both components. Further, by making a simple variable transformation, they were able to express a meaningful bivariate Weibull distribution.

In a recent book, Mardia (47) provides an excellent summary and bibliography of most of the well known bivariate distributions derived up through 1970. Ord (55) provides a similar summary on families of frequency distributions which include certain classes of

bivariate distributions. A summary of the more classical bivariate
distributions given by these two authors is given in Appendix A. More
recent contributions include a derivation of a compound gamma
bivariate by Hutchinson (42) in 1981 and a new class of bivariate
logistic distributions by Ali and Mikhail (21) in 1978. Also, current
interests have extended certain known bivariate distributions into the
complex space. Recent articles include those by Brock and Krutchoff
(25), Giri (36), and Saxena (58).

With the exception of the bivariate normal distribution, most
of the work in multivariate analysis has been in the area of
characterizing a given bivariate distribution either through a
thorough study of its marginal and conditional probability
distributions, or through a study of its characteristic function.
Such analysis is reproduced in books by Springer (17), Anderson (23),
Feller (34,35), Mood and Graybill (52), and Parzen (56). In addition,
recent articles on the subject include those of Lukacs and Beer (46)
and Abrahams and Thomas (20).

Work on the actual distribution of algebraic combinations of
dependent random variables has been mostly limited to the bivariate
normal distribution. In his book, Springer (17) defines the bivariate
normal distribution as well as an established methodology for deriving
the sums of bivariate normal variates using double Fourier integral
transform techniques. The distribution of the product of two
dependent normal variates was studied by Aroian (24) in 1947, and the

distribution of the quotient of two dependent normal variates by Craig (27) in 1942. Nicholson (54) used a geometrical approach to study the ratio between two dependent variables, but was able to apply his results only to the bivariate normal.

Along with new interests in studying and developing new and meaningful bivariate distributions came revived interests in studying products, quotients, and sums of the dependent variates of these distributions. Current research is very specific in nature in that techniques used for one bivariate functional do not apply to another bivariate functional. Indeed, the techniques for products or quotients may not even be the same for a given bivariate density. Current research in this area include Abrahams and Thomas (20), Alsina and Bonet (22), Gupta (38), Lee, Holland, and Flueck (45), Mathai (50), Tan (60), and Wallgren (61). From these studies it becomes clear that a more general theory that is applicable to a wide variety of cases is a much needed tool for the multivariate analyst.

In 1944, Reed (16) defined the double Mellin integral transform and its inversion integral with associated theorems for each. He also presented a theorem for deriving the double Mellin transform of the product of two bivariate functionals. As an example of his theorems, Reed derived the double Mellin transform identities for Appell's hypergeometric functions of two variables.

Fox (10) provided the first practical method of handling products of bivariate density variates by extending the results

derived by Reed (16) to cases of statistical distributions. He also included some work on quotients of bivariate density variates as well as a detailed discussion on how to handle functionals that resided in other than the first quadrant. Fox's work however, was limited to finding a bivariate density function that resulted from products or quotients of variates of two bivariate density functions.

Subrahamian (19) was the first to provide significant insight as to deriving a univariate density function which resulted from a product or quotient of two dependent variables which share a single bivariate distribution. He used the results of Fox (10) and combined them with earlier results on independent variates by Epstein (9) and Springer and Thompson (18). Subrahamian applied his conclusions to the bivariate normal distribution and derived by this alternative method the same results previously presented by Aroian (24) and Craig (27,28).

As a result of the labors of the authors listed above, today's statisticians have a set of powerful tools in transform theory for handling certain algebraic combinations of independent and dependent variables. However, it would seem a natural and powerful extension to this theory if one could apply certain results of Carter's H-function analysis to bivariate distributions as well. A brief review of work on H-functions of two variables follows.

In the early 1970's, Verma (86,87), Mittal and Gupta (75), and Goyal (69) extended the H-function defined by Fox to a generalized

H-function of two variables. In their book, Mathai and Saxena (14) reproduce a formal definition of the H-function of two variables as well as some important properties and identities for the H-function of two variables. They point out the importance of this function arises from the fact that it contains as special cases H-functions of one variable, G-function of two variables, Whittaker functions of two variables, and Appell's functions of two variables.

The majority of H-function work has been highly theoretical and generally restricted to a few special cases. Most of the articles on the subject are by authors from India and are published in foreign or little known journals, and are not easily accessible to the U.S. researcher. Almost no applications are given in the literature and the few given are for physics and engineering. Due to the notation and the curse of dimensionality, this problem is particularly true for the H-function of two variables. Thus far, the limited work done on bivariate H-functions has been in the area of extension of work from G-functions to H-functions, Argarwal and Singhal (62), and on identifying special cases of the bivariate H-function, Anandani (64). Most of the significant work done on the bivariate H-function lies in the area of solving differential or integral equations. Solutions of dual integral equations by the use of H-functions can be seen in the works of Pathak and Prasad (78) and Saxena (82,83). In 1972, Mittal and Gupta (75) used a generalized function of two variables to solve certain classes of integral equations.

Along with a more general and subsequently more useful definition of the bivariate H-function, Goyal (69) provides some insights on the applicability of taking the Laplace transform of the bivariate H-function. Other work in this area includes Prasad and Maurya (79) and DeAnguio and Kalla (66).

1.3 Bivariate Probability Theory

1.3.1 Definitions:

(35:66-74;56:354-365;52:82-98,198-215;23:60-66;43:4-12)

Let X and Y be jointly distributed random variables having the bivariate density function denoted by $f_{X,Y}(x,y)$ and the bivariate cumulative distribution function by $F_{X,Y}(x,y)$. Then

$$F_{X,Y}(x,y) = P(X \leq x , Y \leq y) \tag{1.1}$$

If we are interested only in the cumulative distribution of X, then it is apparent from (1.1) that

$$P(X \leq x) = F_{X,Y}(x,\infty) \tag{1.2}$$

Therefore, $F_X(x) = F_{X,Y}(x,\infty)$ defines the cumulative distribution function for X and its associated density function is given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy \tag{1.3}$$

Similarly, the density function for y is given by

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx \tag{1.4}$$

The density functions $f_X(x)$ and $f_Y(y)$ are known as the marginal

density functions for the joint density function $f_{X,Y}(x,y)$.

The expectation μ_x and variance σ_x^2 of x, if they exist, are given by

$$\mu_x = E(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{X,Y}(x,y) dxdy \qquad (1.5)$$

and

$$\sigma_x^2 = \text{Var}(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x-\mu_x)^2 f_{X,Y}(x,y)dxdy \qquad (1.6)$$

Similar identities are given for the expectation μ_Y and variance σ_y^2 of Y.

The expectation μ_{xy}, or first product moment of X and Y, is given by

$$\mu_{xy} = E(xy) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{X,Y}(x,y)dxdy \qquad (1.7)$$

Equations (1.5) and (1.6) can be written in a more general form for higher ordered moments of $f_{X,Y}(x,y)$. Let $\alpha_{n1,n2}$ be the n1,n2 ordered moment for $f_{X,Y}(x,y)$, then $\alpha_{n1,n2}$ is given by

$$\alpha_{n1,n2} = E(x^{n1} y^{n2})$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^{n1} y^{n2} f_{X,Y}(x,y)dxdy \qquad (1.8)$$

Then, $\alpha_{1,0} = \mu_x$, $\alpha_{0,1} = \mu_y$, and $\alpha_{1,1} = \mu_{xy}$.

Similarly, the central moments, $\mu_{n1,n2}$, for $f_{X,Y}(x,y)$ are given by

$$\mu_{n1,n2} = E\{[x - E(x)]^{n1}[y - E(y)]^{n2}\} \qquad (1.9)$$

Then, $\mu_{2,0} = \sigma_x^2$, $\mu_{0,2} = \sigma_x^2$, and $\mu_{1,1}$ is referred to as the covariance of X and Y.

Specifically rewritten, the covariance $cov(x,y)$ for X and Y is given by

$$cov(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_x)(y-\mu_y)f_{X,Y}(x,y)dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{X,Y}(x,y)dxdy - \mu_x\mu_y \qquad (1.10)$$

The covariance of X and Y is a measure of the dependency between the two random variables X and Y. The dependency structure of X and Y may also be characterized by the correlation coefficient, $\rho(x,y)$, and is given by

$$\rho(x,y) = cov(x,y)/\sigma_x\sigma_y \qquad (1.11)$$

If X and Y are independent, then $\rho(x,y) = 0$. However, it is not necessarily true that if $\rho(x,y) = 0$, then X and Y are independent.

1.3.2 <u>Properties</u> <u>of</u> <u>Moments</u>: (56:354-365)

Let g(x,y) be a function of X and Y where X and Y are jointly continuous with a joint density function $f_{X,Y}(x,y)$. The expected value of a function of two real variables, $E[g(x,y)]$, is defined as

$$E[g(x,y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy \qquad (1.12)$$

From this definition, the following linearity property for expectations of jointly distributed random variables is derived.

<u>Theorem</u> <u>1.1</u>: If X and Y are real random variables which are jointly distributed by $f_{X,Y}(x,y)$, and if X and Y have finite expectations $E(x)$ and $E(y)$, then the sum X+Y has a finite expectation given as

$$E(x + y) = E(x) + E(y) \qquad (1.13)$$

A similar relation may be found for finding the variance of two jointly distributed random variables.

<u>Theorem</u> <u>1.2</u>: If X and Y are real random variables which are jointly distributed by $f_{X,Y}(x,y)$, and if X and Y have finite variances $Var(x)$ and $Var(y)$ and a finite covariance $Cov(x,y)$, then the sum X+Y has a finite variance given as

$$Var(x+y) = Var(x) + Var(y) + 2Cov(x,y) \qquad (1.14)$$

1.3.3 <u>Moment</u> <u>Generating</u> <u>Functions</u>: (56:354-365;52:200-204)

The joint moment-generating function for a probability density function of two variables $f_{X,Y}(x,y)$ is given by

$$\tilde{F}_{X,Y}(t_1,t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{t_1 x + t_2 y} f_{X,Y}(x,y) \, dx \, dy \qquad (1.15)$$

where t_1 and t_2 are two real numbers for which the double integral exists.

If the integral exists, then the following moments may be found:

$$E(x^n) = \frac{\partial^n}{\partial t_1^n} \tilde{F}_{X,Y}(0,0)$$

$$E(y^n) = \frac{\partial^n}{\partial t_2^n} \tilde{F}_{X,Y}(0,0)$$

$$E(x,y) = \frac{\partial^2}{\partial t_1, \partial t_2} \tilde{F}_{X,Y}(0,0)$$

For central moments, replace x and y by $x - E(x)$ and $y - E(y)$ respectively. Then the following central moments may be found:

$$Var(x) = \frac{\partial^2}{\partial t_1^2} \tilde{F}_{X-E(x),Y-E(y)}(0,0)$$

$$Var(y) = \frac{\partial^2}{\partial t_2^2} \tilde{F}_{X-E(x),Y-E(y)}(0,0)$$

$$\text{Cov}(x,y) = \frac{\partial^2}{\partial t_1, \partial t_2} \tilde{F}_{X-E(x),Y-E(y)}(0,0)$$

Proceeding in a similar manner will yield any degree of moments of higher order.

Properties: If $Z = X + Y$, where X and Y are independent, $\tilde{F}_X(t)$ is the moment generating function of X, and $\tilde{F}_Y(t)$ is the moment generating function of Y, then the moment generating function of $\tilde{F}_Z(t)$ is given as

$$\tilde{F}_Z(t) = \tilde{F}_X(t) \cdot \tilde{F}_Y(t).$$

<u>Characteristic Function</u>: In cases where the integral in (1.15) does not exist, moments may still be found in an identical fashion by use of the characteristic function. The characteristic function is identical to equation (1.15) with t_1 and t_2 replaced by it_1 and it_2 respectively.

1.4 Transformations of Random Variables

Emphasis in this area is on the use of integral transforms to obtain probability density functions for certain transformations of random variables. First, a review of some related probability concepts should be made.

A one-to-one transformation h(x) from a set S into a set T means that for each y, an element of T, there exists one and only one x, an element of S, such that h(x) = y. When a function h(x) is a one-to-one transformation from a set S to a set T, then the inverse transformation $h^{-1}(y)$, from T onto S, exists and $h^{-1}[h(x)] = x$. Stating that a set S in the set of positivity for a transformation h(x) means that S is the set of values, x, for which h(x) is positive.

Two random variables X and Y are independent if their joint probability density function $f_{X,Y}(x,y)$ equals the product of their marginal density functions $f_X(x)$ and $f_Y(y)$. This means that any variation in X will in no way affect the outcome of Y, or vice versa.

Theorem 1.3: Let X be a random variable with continuous probability function $f_X(x)$ and suppose that y = h(x) is a one-to-one transformation from S, the set of positivity of $f_X(x)$, onto T, the image of S under h(x). If $h^{-1}(y)$ is differentiable and its derivative is continuous on T, then the probability density function of Y may be given as

$$f_Y(y) = \begin{cases} f[h^{-1}(y)] \dfrac{d}{dy} h^{-1}(y) & , y \ \varepsilon \ T \\[4mm] 0 & , else \end{cases}$$

__Theorem__ __1.4__: Let $\underline{X} = (X_1, X_2, \ldots X_k)$ be a set of k random variables having the joint continuous probability density function $f_{\underline{X}}(\underline{x})$. Let $\underline{Y} = h(\underline{x}) = \{h_1(\underline{x}), h_2(\underline{x}), \ldots h_k(\underline{x})\}$ be a set of relations forming a one-to-one transformation from S, the k-dimensional set of positivity of $f_{\underline{x}}$, onto T, the k-dimensional image of S under $h(\underline{x})$. The inverse transformation exists, $\underline{X}^{-1} = h^{-1}(\underline{y}) = \{g_1(\underline{y}), g_2(\underline{y}), \ldots g_k(\underline{y})\}$. If the partial derivatives of $h^{-1}(\underline{y})$ exist and are continuous,

$$g_{ij} = \frac{\partial}{\partial y_j} \{g_i(y_1, y_2, \ldots y_k)\} \qquad\qquad i,j = 1,2,\ldots k$$

then the joint probability density function of $\underline{Y}$ is given by

$$f_{\underline{Y}}(\underline{y}) = \begin{cases} f_{\underline{X}}\{g_1(\underline{y}), g_2(\underline{y}), \ldots g_k(\underline{y})\} \left| J \right| & , y \ \varepsilon \ T \\[4mm] 0 & , else \end{cases}$$

where J is the Jacobian, the determinant of first partial derivatives,

$$J = \begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1k} \\ g_{21} & g_{22} & \cdots & g_{2k} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ g_{k1} & g_{k2} & \cdots & g_{kk} \end{vmatrix}$$

Using Theorem 1.4, one can find the distributions for the sum, product, difference, and quotient of two jointly distributed random variables.

Example 1.1: Suppose the probability density function of $Z = X/Y$ is desired. Let $W = Y$ so that $X = ZW$ and $Y = W$ and

$$J = \begin{vmatrix} \partial x/\partial z & \partial x/\partial w \\ \partial y/\partial z & \partial y/\partial w \end{vmatrix} = \begin{vmatrix} w & z \\ 0 & 1 \end{vmatrix} = w$$

By Theorem 1.4, for the approprate ranges of z and w,

$$f_{Z,W}(z,w) = f_{X,Y}(zw,w) \left| J \right|$$

The marginal distribution of $Z = X/Y$ is found by integrating the above joint distribution $f_{Z,W}(z,w)$ over the proper range of w:

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(zy,y)y\,dy$$

Notice that if $W = X$ so that $Y = W/Z$ then the determinant of the

Jacobian equals $-w/z^2$. Then

$$f_Z(z) = \int_{-\infty}^{\infty} (x/z^2)f_{X,Y}(x,x/z)dx$$

The point here is that the two forms for calculating $f_Z(z)$ are not identical by a simple interchange of the x and y variables. This is the only case where the symmetry does not hold. While both forms are certainly valid, the first form is the one most commonly seen in the literature and is usually the most easily applied.

Using Theorem 1.4 similarly to find the distributions for the difference, product, and sums of two random variables gives the following theorem (41).

Theorem 1.5: If X and Y are jointly continuous random variables with probability density function $f_{X,Y}(x,y)$, then

(1) the probability density function of the random variable Z = X + Y is given by

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(z-y,y)dy = \int_{-\infty}^{\infty} f_{X,Y}(x,z-x)dx$$

If X and Y are independent, then

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y)f_Y(y)dy = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx$$

(2) the probability density function of the random variable Z = X - Y is given by

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(z+y,y)dy = \int_{-\infty}^{\infty} f_{X,Y}(x,z+x)dx$$

If X and Y are independent, then

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z+y)f_Y(y)dy = \int_{-\infty}^{\infty} f_X(x)f_Y(z+x)dx$$

(3) the probability density function of the random variable Z = XY is given by

$$f_Z(z) = \int_{-\infty}^{\infty} |y^{-1}| f_{X,Y}(z/y,y)dy = \int_{-\infty}^{\infty} |x^{-1}| f_{X,Y}(x,z/x)dx$$

If X and Y are independent, then

$$f_Z(z) = \int_{-\infty}^{\infty} |y^{-1}| f_X(z/y)f_Y(y)dy = \int_{-\infty}^{\infty} |x^{-1}| f_X(x)f_Y(z/x)dx$$

(4) the probability density function of the random variable Z = X/Y is given by

$$f_Z(z) = \int_{-\infty}^{\infty} |y| f_{X,Y}(zy,y)dy = \int_{-\infty}^{\infty} \left|\frac{x}{z^2}\right| f_{X,Y}(x,\frac{x}{z})dx$$

If X and Y are independent, then

$$f_Z(z) = \int_{-\infty}^{\infty} |y| f_X(zy)f_Y(y)dy = \int_{-\infty}^{\infty} \left|\frac{x}{z^2}\right| f_X(x)f_Y(\frac{x}{z})dx.$$

Theorems 1.4 and 1.5 have been applied to many distribution problems. However, each case must be treated separately and special care must be taken to determine the proper integration limits and ranges for the variables. A look at some simple examples can help to clarify this point.

Example 1.2: Consider the bivariate standard normal given by

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right\}$$

$$-\infty < x,y < \infty, \quad \rho \neq 1,-1$$

Suppose we wish to find $f_Z(z)$ where $Z = X/Y$. Then

$$f_Z(z) = \int_{-\infty}^{\infty} |x| f_{X,Y}(x,zx)dx$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left\{\frac{-1}{2(1-\rho^2)}(x^2 - \rho zx^2 + z^2x^2)\right\}|x|dx$$

Let $a = (1 - 2\rho z + z^2)/2(1-\rho^2)$. Then

$$f_Z(z) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} x e^{-ax^2}dx$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \frac{1}{a}$$

$$= \frac{\sqrt{1-\rho^2}}{\pi(1 - 2\rho z + z^2)}$$

This is equivalent to Craig's (28) derivation if $\sigma_x = \sigma_y = 1$ and $\mu_x = \mu_y = 0$.

Example 1.3: Now consider Morgensterm's bivariate uniform density given by

$$f_{X,Y}(x,y) = 1 + \rho(2x - 1)(2y - 1) \qquad 0 \le x,y \le 1$$

Let $Z = XY$. Then

$$f_Z(z) = \int_{-\infty}^{\infty} \left| y^{-1} \right| f_{X,Y}(z/y,y)dy$$

The lower limit of integration of y is determined from the relation

$$x = z/y, \quad 0 \le x \le 1, \quad 0 \le y \le 1$$

Since $x \le 1$, the lower limit on y is z. Furthermore, since $x \le 1$ and $y \le 1$, the maximum value of $z = xy$ is 1. Then

$$f_Z(z) = \int_z^1 (1/y)[1 + \rho(2z/y - 1)(2y - 1)dy$$

$$= [\rho(4z + 1) + 1]\ln(1/z) + 4\rho(z - 1) \qquad , \; 0 < z \le 1$$

If $\rho = 0$, then X and Y are independent and $f_Z(z)$ reduces to

$$f_Z(z) = -\ln(z) \qquad , \; 0 < z \le 1$$

This is the same form as that derived by Springer (17:91-94) for the case of the product of two independent identically distributed uniform variates.

Example **1.4**: Suppose the probability density function of $Z = XY$ is desired where X and Y are jointly distributed by Kellogg-Barnes III distribution given as

$$f_{X,Y}(x,y) = \frac{\beta \alpha^2}{\Gamma(c)} x^c \, e^{-(\alpha x + \beta xy)} \qquad \begin{array}{l} x,y>0 \\ \alpha,\beta>0 \ , \ c>2 \end{array}$$

from Theorem 1.5, case (3), the distribution for Z is given by

$$f_Z(z) = \int_0^\infty \frac{1}{x} f_{X,Y}(x, z/x) dx$$

$$= \frac{\beta \alpha^c}{\Gamma(c)} e^{-\beta z} \int_0^\infty x^{c-1} e^{-\alpha x} \, dx$$

$$= \beta e^{-\beta z} \qquad z>0$$

which is the univariate exponential distribution with parameter $1/\beta$.

Example **1.5**: Consider the Kellogg-Barnes II distribution given by

$$f_{X,Y}(x,y) = \beta \alpha^2 \, e^{-(\alpha x + \beta y/x)} \qquad \begin{array}{l} x,y>0 \\ \alpha,\beta>0 \end{array}$$

From Theorem 1.5, case (4), the distribution of the random variable $Z = X/Y$ may be found by

$$f_Z(z) = \int_0^\infty y f_{X,Y}(zy,y) dy$$

$$= \beta\alpha^2 e^{-\beta/z} \int_0^\infty y e^{-\alpha z y} dy$$

$$= (\beta/z^2) e^{-\beta/z}$$

Now suppose the reverse is desired, that is, the distribution of $Z = Y/X$. Theorem 1.5, case (4), still applies by using a simple change of variables and $f_Z(z)$ is given by

$$f_Z(z) = \int_0^\infty x f_{X,Y}(x,zx) dx$$

$$= \beta\alpha^2 e^{-\beta z} \int_0^\infty x e^{-\alpha x} dx$$

$$= \beta e^{-\beta z}$$

While Examples 1.4 and 1.5 are relatively straight forward, Examples 1.2 and 1.3 show that the use of Theorem 1.5 for products and quotients of dependent variables can be an arduous task. For products and quotients of dependent variables the task can be simplified through the use of Mellin transform techniques as will be shown in the next chapter.

CHAPTER 2

Application of Integral Transforms to Statistical Analysis

2.1 General Remarks

Section 1.4 of Chapter 1 shows that using convolution integrals for the transformation of variables to find the distribution for the sum, difference, product, or quotient of two random variates can be a difficult task. This chapter outlines techniques to simplify the problem by utilizing integral transform techniques. A review of integral transforms and the associated techniques for finding distributions of algebraic combinations of independent variates is presented followed by a discussion of the extension to products and quotients of dependent variates using double Mellin transform techniques.

Since this dissertation is devoted to products and quotients of dependent variates using double Mellin transform techniques, the double Mellin transform is developed more completely. Theorems governing its use as given by Fox (10) and Reed (16) are presented. Extensions to univariate Mellin transform properties are presented for the double Mellin transform. These properties prove useful to Mellin transform manipulations in later chapters.

Finally, theorems are given for the distribution of products, quotients, and rational powers of two dependent variates. Theorems are also presented for products, quotients, and powers of variates from two bivariate distributions which are pairwise independent.

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2.2 Integral Transforms

2.2.1 Double Fourier Transform: (17:67-75;106:76-79)

A real function of two variables $f_{X,Y}(x,y)$, where each variable is defined over the whole real line, is doubly Fourier transformable if the integral

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| f_{X,Y}(x,y) \right| e^{ik_1x + ik_2y} \, dxdy$$

converges for some real value for k_1 and k_2. Then,

$$F_{t_1,t_2}\{f_{X,Y}(x,y)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{it_1x + it_2y} f_{X,Y}(x,y) \, dxdy \qquad (2.1)$$

is the double Fourier transform of $f_{X,Y}(x,y)$. $F_{t_1,t_2}\{f_{X,Y}(x,y)\}$ is called the bivariate characteristic function of $f_{X,Y}(x,y)$, and $e^{it_1x + it_2y}$ is called the kernel. The inversion integral is given by

$$f_{X,Y}(x,y) = \frac{1}{2\pi} \int_{h-i\infty}^{h+i\infty} \int_{k-i\infty}^{k+i\infty} e^{-it_1x - it_2y} F(t_1,t_2) \, dt_1 dt_2 \qquad (2.2)$$

2.2.2 Double Laplace Transform: (106:221-228;34:452-458)

A real function of two variables $f_{X,Y}(x,y)$, defined everywhere for $x > 0$ and $y > 0$, with x and y real, is doubly Laplace transformable if the integral

$$\int_0^\infty \int_0^\infty e^{-k_1 x - k_2 y} \left| f_{X,Y}(x,y) \right| dx dy$$

converges for some real values of k_1 and k_2. Then

$$L_{r_1, r_2} \{ f_{X,Y}(x,y) \} = \int_0^\infty \int_0^\infty e^{-r_1 x - r_2 y} f_{X,Y}(x,y) dx dy \qquad (2.3)$$

is the double Laplace transform of $f_{X,Y}(x,y)$, where r_1 and r_2 are complex variables.

From this definition it can be shown that

$$L_{r_1, r_2} \{ f_{X,Y}(ax,by) \} = (ab)^{-1} L(r_1/a, r_2/b) \qquad (2.4)$$

The double Laplace transform has been used in the past for functions of x and y where x and y are independent. Only limited consideration has been given to dependent functions of x and y.

2.2.3 Double Mellin Transform: (17:151-156;10;16;19)

A real function of two variables $f_{X,Y}(x,y)$, defined everywhere for $x > 0$ and $y > 0$, with x and y real, has been defined by Reed (13) to have the double Mellin integral transform given as

$$M(s_1, s_2) = \int_0^\infty \int_0^\infty x^{s_1 - 1} y^{s_2 - 1} f_{X,Y}(x,y) dx dy \qquad (2.5)$$

and its inverse as

$$f_{X,Y}(x,y) = \frac{1}{(2\pi i)^2} \int_{h-i\infty}^{h+i\infty} \int_{k-i\infty}^{k+i\infty} x^{-s_1} y^{-s_2} M(s_1,s_2) ds_1 ds_2 \qquad (2.6)$$

The conditions for which (2.5) and (2.6) are valid are stated by the following theorems. The proofs are given by Reed (16) and Fox (10).

Theorem 2.1: If

(i) $M(s_1,s_2)$ is a regular function of both variables s_1, s_2 in the strips $a < s_1 < b$, $c < s_2 < d$

(ii) in these strips $M(s_1,s_2) = O(|s_1|^{-m}) O(|s_2|^{-n})$ for some $m > 0$, $n > 0$, as $|s_1|$ and $|s_2|$ tend to infinity independently of each other;

(iii) $a < h < b$ and $c < k < d$

(iv) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |M(s_1,s_2)| |ds_1| |ds_2|$ exists when taken along any lines parallel to the imaginary axis in the strips defined in (i)

(v) $f_{X,Y}(x,y)$ is defined by equation (2.6)

then

$$M(s_1,s_2) = \int_0^{\infty} \int_0^{\infty} x^{s_1-1} y^{s_2-1} f_{X,Y}(x,y) dx dy$$

is true.

Theorem 2.2: Let X denote a part of the complex x plane which is bounded by two lines through the origin and which includes the whole of the positive real axis from 0 to $+\infty$. Let Y denote a similar region in the complex y plane. If with x in X and y in Y the following conditions are satisfied:

(i) there exists two real numbers h and k such that $x^h y^k f_{X,Y}(x,y)$ is a regular function of both x and y;

(ii) $x^h y^k f_{X,Y}(x,y) = O(|\log x|^{-m})O(|\log y|^{-n})$, $m > 0$, $n > 0$, as x and y tend to infinity independently

(iii) $\iint |x^h y^k f_{X,Y}(x,y)| |dx| |dy|$ exists, when taken along any lines in the X and Y regions

(iv) $M(s_1,s_2)$ is defined by equation (2.5)

then equation (2.6) is true

These two theorems give an exact analogue to the single Mellin integral transform theorems.

2.2.4 Mellin Transform Properties

By making the change of variables $x = u/a$ and $y = v/b$ in the defining integral (2.5),

$$M_{s_1,s_2}\{ f(ax,by) \} = \int_0^\infty \int_0^\infty f(u,v) u^{s_1-1} v^{s_2-1} ds_1 ds_2$$

$$= a^{-s_1} b^{-s_2} M_{s_1,s_2}\{ f(u,v) \} \tag{2.7}$$

Since multiplying $f(x,y)$ by $x^a y^b$ merely results in changing s_1

to s_1+a and s_2 to s_2+b, we also have the following relation:

$$M_{s_1,s_2}\{ x^a y^b f(x,y) \} = M_{s_1+a,s_2+b}\{ f(x,y) \} \qquad (2.8)$$

For $a,b > 0$, making the change of variables $x=u^{1/a}$ and $y=v^{1/b}$ yields

$$M_{s_1,s_2}\{f(x^a,y^b)\} = \int_0^\infty \int_0^\infty f(u,v) u^{s_1/a-1/a} v^{s_2/b-1/b} ((ab)^{-1} u^{1/a-1} v^{1/b-1} du dv)$$

$$= (ab)^{-1} \int_0^\infty \int_0^\infty f(u,v) u^{s_1/a-1} v^{s_2/b-1} du dv$$

$$= (ab)^{-1} M_{s_1/a,s_2/b}\{ f(u,v) \} \qquad (2.9)$$

The Mellin transform also has certain unique properties for derivatives and integrals of functions. By definition

$$M_{s_1,s_2}\{ \frac{\partial}{\partial x} f(x,y) \} = \int_0^\infty \int_0^\infty x^{s_1-1} y^{s_2-1} [\frac{\partial}{\partial x} f(x,y)] dx dy$$

If $[(\partial/\partial x)f(x,y)]$ is continuous for y constant and $0 < y < \infty$ then the equation above can be written (89;92:179-180)

$$M_{s_1,s_2}\{ \frac{\partial}{\partial x} f(x,y) \} = \int_0^\infty y^{s_2-1} \int_0^\infty x^{s_1-1} \frac{\partial}{\partial x} f(x,y) dx dy$$

integrating the inner integrand by parts yields

$$M_{s_1,s_2}\{\frac{\partial}{\partial x}f(x,y)\} = \int_0^\infty y^{s_2-1}\left[f(x,y)x^{s_1-1}\Big|_0^\infty -(s_1-1) \int_0^\infty x^{s_1-2} f(x,y)dx\right]dy$$

If there exist σ_1, σ_2 such that

$$\lim_{x \to 0} x^{s_1-1} f(x,y) = 0 \quad ; \quad \lim_{x \to \infty} x^{s_1-1} f(x,y) = 0$$

when $\sigma_1 < \mathrm{Re}\ s_1 < \sigma_2$ and if $M_{s_1-1,s_2}\{ f(x,y) \}$ exists in that band, then

$$M_{s_1,s_2}\{ \frac{\partial}{\partial x} f(x,y) \} = -(s_1-1)M_{s_1-1,s_2}\{ f(x,y) \} \qquad (2.10)$$

Equation (2.10) can be written conveniently in terms of integrals rather than derivatives. Equation (2.10) can be written in the form

$$M_{s_1,s_2}\{ f(x,y) \} = (s_1-1)M_{s_1-1,s_2}\{ \int_x^{\infty} f(u,y)du \}$$

Replacing s_1 by s_1+1 yields

$$M_{s_1,s_2}\{ \int_x^{\infty} f(u,y)du \} = s_1^{-1} M_{s_1+1,s_2}\{ f(x,y) \} \qquad (2.11)$$

Similarly, if $[(\partial/\partial y)f(x,y)]$ is continuous for x constant, $0 < x < \infty$ and if there exist σ_3, σ_4 such that

$$\lim_{y \to 0} y^{s_2-1} f(x,y) = 0 \quad ; \quad \lim_{y \to \infty} y^{s_2-1} f(x,y) = 0$$

when $\sigma_3 < \mathrm{Re}\ s_2 < \sigma_4$ and if $M_{s_1,s_2-1}\{ f(x,y) \}$ exists in that band,

then the following hold.

$$M_{s_1,s_2}\{ \frac{\partial}{\partial x} f(x,y) \} = -(s_2-1)M_{s_1,s_2-1}\{ f(x,y) \} \qquad (2.12)$$

$$M_{s_1,s_2}\{ \int_y^\infty f(x,v)dv \} = s_2^{-1} M_{s_1,s_2+1}\{ f(x,y) \} \qquad (2.13)$$

Higher derivatives can be dealt with in a similar fashion. Applying (2.11) and (2.13) iteratively, the following property for a double integral can be written.

$$M_{s_1,s_2}\{ \int_x^\infty \int_y^\infty f(u,v)dudv \} = (s_1s_2)^{-1} M_{s_1+1,s_2+1}\{ f(x,y) \} \qquad (2.14)$$

Equation (2.14) above is a useful property in that it may now be used in developing the cumulative of the H-function distribution.

The properties developed above are summarized below.

$$M_{s_1,s_2}\{ f(ax,by) \} = a^{-s_1} b^{-s_2} M_{s_1,s_2}\{ f(x,y) \}$$

$$M_{s_1,s_2}\{ x^a y^b f(x,y) \} = M_{s_1+a,s_2+b}\{ f(x,y) \}$$

$$M_{s_1,s_2}\{ f(x^a,y^b) \} = (ab)^{-1} M_{s_1/a,s_2/b}\{ f(x,y) \} \quad ; \quad a,b > 0$$

$$M_{s_1,s_2}\{ \int_x^\infty \int_y^\infty f(u,v)dudv \} = (s_1s_2)^{-1} M_{s_1+1,s_2+1}\{ f(x,y) \}$$

2.2.5 <u>Mellin</u> <u>Transform</u> <u>of</u> <u>Appell's</u> <u>Functions</u>: (16;8:232)

Reed (16) used the double Mellin integral transform to obtain

the following transforms for Appells hypergeometric functions of two variables:

$$\int_0^\infty \int_0^\infty x^{s_1-1} y^{s_2-1} F_1(a,b,b';c;-x,-y)dxdy$$

$$= \frac{\Gamma(c)\Gamma(s_1)\Gamma(s_2)\Gamma(a-s_1-s_2)\Gamma(b-s_1)\Gamma(b'-s_2)}{\Gamma(a)\Gamma(b)\Gamma(b')\Gamma(c-s_1-s_2)} \qquad (2.15)$$

when $0 < Re(s_1+s_2) < Re(a)$, $0 < Re(s_1) < Re(b)$, $0 < Re(s_2) < Re(b')$

$$\int_0^\infty \int_0^\infty x^{s_1-1} y^{s_2-1} F_2(a,b,b';c,c';-x,-y)dxdy$$

$$= \frac{\Gamma(s_1)\Gamma(s_2)\Gamma(c)\Gamma(c')\Gamma(a-s_1-s_2)\Gamma(b-s_1)\Gamma(b'-s_2)}{\Gamma(a)\Gamma(b)\Gamma(b')\Gamma(c-s_1)\Gamma(c'-s_2)} \qquad (2.16)$$

when $0 < Re(s_1+s_2) < Re(a)$, $0 < Re(s_1) < Re(b)$, $0 < Re(s_2) < Re(b')$

$$\int_0^\infty \int_0^\infty x^{s_1-1} y^{s_2-1} F_3(a,a',b,b';c;-x,-y)dx\,dy$$

$$= \frac{\Gamma(s_1)\Gamma(s_2)\Gamma(c)\Gamma(a-s_1)\Gamma(a'-s_2)\Gamma(b-s_1)\Gamma(b'-s_2)}{\Gamma(a)\Gamma(a')\Gamma(b)\Gamma(b')\Gamma(c-s_1-s_2)} \qquad (2.17)$$

when $0 < Re(s_1) < \min(Re(a), Re(b))$, $0 < Re(s_2) < \min(Re(a'),Re(b'))$

$$\int_0^\infty \int_0^\infty x^{s_1-1} y^{s_2-1} F_4(a,b;c,c';-x,-y)dxdy$$

$$= \frac{\Gamma(s_1)\Gamma(s_2)\Gamma(c)\Gamma(c')\Gamma(a-s_1-s_2)\Gamma(b-s_1-s_2)}{\Gamma(a)\Gamma(b)\Gamma(c-s_1)\Gamma(c'-s_2)} \tag{2.18}$$

when $0 < Re(s_1+s_2) < Re(a)$, $0 < Re(s_1+s_2) < Re(b)$.

Equations (2.15) - (2.18) above may also be derived by making appropriate sign changes in the formulas given by Erdelyi (8:232). The importance of these transform identities will be shown later in Chapter 3 when special cases of the bivariate H-function are identified.

2.3 Integral Transforms for Independent Variates

From the examples it can be seen that using Theorem 1.5 can be a difficult process. Integral transforms can help simplify the process. For the case of X and Y independent, the following formulas have been most helpful in solving algebraic combinations of random variables.

Let F_t, L_r, and M_s represent the Fourier, Laplace, and Mellin integral transforms of one variable respectively. Then special properties of these transforms are

$$F_t\{f_X(x)\} \; F_t\{g_Y(y)\} = F_t\{ \int_{-\infty}^\infty f_X(x)g_Y(y-x)dx \}$$

$$L_\tau\{f_X(x)\} \ L_\tau\{g_Y(y)\} = L_\tau\{ \int_{-\infty}^{\infty} f_X(x)g_Y(y-x)dx \}$$

$$M_s\{f_X(x)\} \ M_s\{g_Y(y)\} = M_s\{ \int_{0}^{\infty} x^{-1} f_X(x)g_Y(y/x)dx \}$$

Combining these formulas with Theorem 1.5, the following results can be derived:

(1) the probability density function of the random variable Z = X+Y is given by

$$f_Z(z) = F_1^{-1}\{ \ F_t[f_X(x)] \ F_t[f_Y(y)] \ \}$$

or

$$f_Z(z) = L_1^{-1}\{ \ L_\tau[f_X(x)] \ L_\tau[f_Y(y)] \ \} \quad x,y,z \geq 0$$

where F_1^{-1} and L_1^{-1} are the inverse Fourier and Laplace transforms of one variable respectively.

(2) the probability density function of the random variable Z = X-Y is given by

$$f_Z(z) = F_1^{-1}\{ \ F_t[f_X(x)] \ F_t[f_Y(-y)] \ \}$$

where F_1^{-1} is the inverse Fourier transform of one variable.

(3) the probability density function of the random variable Z = XY is given by

$$f_Z(z) = M_1^{-1}\{ \ M_s[f_X(x)] \ M_s[f_Y(y)] \ \} \quad x,y,z \geq 0$$

where M_1^{-1} is the inverse Mellin transform of one variable.

(4) the probability density function of the random variable $Z = X/Y$ is given by

$$f_Z(z) = M_1^{-1}\{ M_s[f_X(x)] M_{2-s}[f_Y(y)] \} \quad x,y,z \geq 0$$

where M_1^{-1} is the inverse Mellin transform of one variable.

A distinct advantage to transform techniques is that these above formulas can be easily extended to more than two variables. However, these formulas are restricted to cases where X and Y are independent.

2.4 Mellin Transforms for Dependent Variates

In 1957, Fox (10) applied the double Mellin integral transform to the theory of bivariate statistics and derived the following conclusions for relations of two bivariate distributions.

The expectation of $\phi(x,y)$, $E[\phi(x,y)]$, is defined by (31:260-265)

$$E[\phi(x,y)] = \int_0^\infty \int_0^\infty \phi(x,y)f(x,y) \, dxdy$$

If $\phi(x,y) = x^{s_1-1} y^{s_2-1}$, then $E[\phi(x,y)]$ is the double Mellin transform definition given by (2.5). Let $f_{X_1,Y_1}(x_1,y_1)$ and $f_{X_2,Y_2}(x_2,y_2)$ be two bivariate density functions having the double Mellin transforms $M_1(s_1,s_2)$ and $M_2(s_1,s_2)$ respectively, and where x_1, x_2, y_1, $y_2 > 0$. Further, assume X_1,Y_1 are independent of X_2,Y_2. Then

$$E[\phi_1(x_1,y_2)\phi_2(x_2,y_2)] = E[\phi_1(x_1,y_1)]\, E[\phi_2(x_2,y_2)]$$

since X_1, Y_1 is pairwise independent of X_2, Y_2. If $\phi_1(x_1,y_1) = x_1^{s_1-1} y_1^{s_2-1}$ and $\phi(x_2,y_2) = x_2^{s_1-1} y_2^{s_2-1}$, then

$$E[\, (x_1 x_2)^{s_1-1} (y_1 y_2)^{s_2-1} \,] = M_1(s_1,s_2)\, M_2(s_1,s_2)$$

The joint probability density function of $(x_1 x_2, y_1 y_2)$ can now be solved by substituting $E[(x_1 x_2)^{s_1-1} (y_1 y_2)^{s_2-1}]$ into the right hand double integral of (2.6):

$$f_{Z,W}(z,w) = M_2^{-1}[\, M_1(s_1,s_2) M_2(s_1,s_2) \,] \tag{2.19}$$

where $Z = X_1 X_2$, $W = Y_1 Y_2$, and M_2^{-1} is the double Mellin inversion transform as defined by (2.6). Continuing in this fashion, Fox showed the following to be true.

$$f_{Z,W}(z,w) = M_2^{-1}[\, M_1(2-s_1,s_2) \,] \tag{2.20}$$

for $Z = 1/X_1$ and $W = Y_1$

$$f_{Z,W}(z,w) = M_2^{-1}[\, M_1(2-s_1,2-s_2) \,] \tag{2.21}$$

for $Z = 1/X_1$ and $W = 1/Y_1$

$$f_{Z,W}(z,w) = M_2^{-1} [M_1(s_1,s_2)M_2(2-s_1,2-s_2)] \qquad (2.22)$$

for $Z = X_1/X_2$ and $W = Y_1/Y_2$

Subrahamian (19) combined the work of Fox (10) to derive results for cases of products and quotients of two dependent variables. Let X and Y be dependent random variables with probability density function $f_{X,Y}(x,y)$ which is positive in the first quadrant and zero elsewhere. Further, suppose that the double Mellin transform of $f_{X,Y}(x,y)$ exists and is given by $M(s_1,s_2)$. If $\phi(x,y) = x^{s-1}y^{s-1}$ then

$$E[x^{s-1}y^{s-1}] = \int_0^\infty \int_0^\infty x^{s-1}y^{s-1} f_{X,Y}(x,y) \, dxdy$$

$$= M_{s,s}\{ f_{X,Y}(x,y) \}$$

Let $Z = XY$, then

$$E[x^{s-1}y^{s-1}] = E[z^{s-1}]$$

Substituting $E[z^{s-1}]$ into the inversion integral for the univariate Mellin transform, the density function for Z can be found. Specifically,

$$f_Z(z) = M_1^{-1}[M(s,s)] \qquad (2.23)$$

where $Z = XY$ and M_1^{-1} is the univariate Mellin transform inversion operator.

Similarly, the probability density function of the random variable Z = X/Y is given by

$$f_Z(z) = M_1^{-1}[\ M(s,2-s)\ ] \tag{2.24}$$

The work accomplished by Fox and Subrahamian can be extended to n pairs of dependent variables which are pairwise independent. This work, shown in the following theorems, is similar to that done by Carter (3) for independent variables. Theorems extending Subrahamian's work to raise dependent variates to rational powers are also given.

Theorem 2.3: If X and Y are dependent continuous random variables with a bivariate probability density function $f_{X,Y}(x,y)$, x,y > 0, then the bivariate probability density function of $Z = X^a$ and $W = Y^b$, a,b rational, is given by

$$f_{Z,W}(z,w) = \begin{cases} M_2^{-1}\ M_{as_1-a+1,bs_2-b+1}\{\ f_{X,Y}(x,y)\ \} & z,w > 0 \\ \\ 0 & \text{otherwise} \end{cases} \tag{2.25}$$

Proof of Theorem 2.3

The Mellin transform of the probability density function $f_{Z,W}(z,w)$ is given by

$$M_{s_1,s_2}\{ f_{Z,W}(z,w) \} = \int_0^\infty \int_0^\infty f_{Z,W}(z,w) z^{s_1-1} w^{s_2-1} \, dz\,dw$$

$$= E[\, z^{s_1-1} w^{s_2-1} \,]$$

where E is the expected value operator. From the definition of Z and W, this becomes

$$M_{s_1,s_2}\{ f_{Z,W}(z,w) \} = E[\, (x^a)^{s_1-1} (y^b)^{s_2-1} \,]$$

$$= \int_0^\infty \int_0^\infty f_{X,Y}(x,y) x^{as_1-a} y^{bs_2-b} \, ds_1\,ds_2$$

$$= \int_0^\infty \int_0^\infty f_{X,Y}(x,y) x^{(as_1-a+1)-1} y^{(bs_2-b+1)-1} \, ds_1\,ds_2$$

$$= M_{as_1-a+1,\,bs_2-b+1}\{ f_{X,Y}(x,y) \}$$

From the inverse Mellin transform, (2.25) follows.

Theorem 2.4: If X and Y are dependent continuous random variables with a bivariate probability density function $f_{X,Y}(x,y)$, $x,y > 0$, then the probability density function of the random variable $Z = X^a Y^b$, a,b rational, is given by

$$f_Z(z) = \begin{cases} M_1^{-1} \ M_{as-a+1,\,bs-b+1}\{ f_{X,Y}(x,y) \} & z > 0 \\ \\ 0 & \text{otherwise} \end{cases} \qquad (2.26)$$

Proof of Theorem 2.4

Let $U = X^a$ and $V = Y^b$. From Theorem 2.3

$$f_{U,V}(u,v) = M_2^{-1}[\ M_{as_1-a+1,bs_2-b+1}\{\ f_{X,Y}(x,y)\ \}\ ]$$

from which it follows that

$$M_{s_1,s_2}\{\ f_{U,V}(u,v)\ \} = M_{as_1-a+1,bs_2-b+1}\{\ f_{X,Y}(x,y)\ \}$$

Since $Z = UV$, applying Equation (2.24) yields

$$f_Z(z) = M_1^{-1}[\ M_{s,s}\{\ f_{U,V}(u,v)\ \}\ ]$$

$$= M_1^{-1}[\ M_{as-a+1,bs-b+1}\{\ f_{X,Y}(x,y)\ \}\ ]$$

Example 2.1: (19;17:154-156)

Consider the bivariate standard normal distribution. For $x,y \geq 0$, the double Mellin transform is

$$M_{X,Y}(s_1,s_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_0^\infty \int_0^\infty x^{s_1-1}\ y^{s_2-1}\ \exp\left\{\frac{-1}{2(1-\rho^2)}(x^2-\rho xy-y^2)\right\} dxdy$$

$$= \frac{\Gamma(s_1)\Gamma(s_2)(1-\rho^2)^{[(s_1+s_2+1)/2-1]}}{2^{(s_1+s_2+1)/2}\ \Gamma(1/2)\Gamma[(s_1+s_2+1)/2]}\ {}_2F_1\left[\ \frac{s_1}{2};\frac{s_2}{2};\frac{s_1+s_2+1}{2};1-\rho^2\right]$$

From Theorem 2.4, for $Z = X/Y$, $a = 1$ and $b = -1$. The Mellin transform of $f_Z(z)$ is then given by

$$M_Z(s) = M_{s,2-s}\{\ f_{X,Y}(x,y)\ \}$$

$$= \frac{\Gamma(s)\Gamma(2-s)}{2\Gamma(1/2)\Gamma(3/2)} \sqrt{(1-\rho^2)} \ _2F_1\left[\ \frac{s}{2}\ ;\ \frac{2-s}{2}\ ;\ \frac{3}{2}\ ;\ 1-\rho^2\right]$$

The transform above is valid for $z > 0$ only, but by symmetry, the inverse for $z < 0$ may be derived also. Subrahamian completed the inversion to give

$$f_Z(z) = \frac{\sqrt{1 - \rho^2}}{\pi(1 - 2\rho z + z^2)}$$

This is exactly the form derived in Example 1.2 for $\sigma_x = \sigma_y = 1$, and $\mu_x = \mu_y = 0$.

Example 2.2: Consider again, Morgenstern's bivariate uniform density.

$$f_{X,Y}(x,y) = 1 + \rho(2x - 1)(2y - 1) \qquad 0 \le x,y \le 1$$

The double Mellin transform of this density is given by

$$M_{X,Y}(s_1,s_2) = \int\limits_{0}^{\infty}\int\limits_{0}^{\infty} x^{s_1-1}\ y^{s_2-1}\ [1 + \rho(2x - 1)(2y - 1)]dxdy$$

$$= \frac{1}{s_1 s_2} + \frac{\rho(s_1 - 1)(s_2 - 1)}{s_1 s_2(s_1 + 1)(s_2 + 1)}$$

If one is interested in the density function $f_Z(z)$, where $Z = XY$, then from Theorem 2.4 $a = b = 1$. The Mellin transform of $f_z(z)$ is then

$$M_Z(s) = M_{s,s}\{ f_{X,Y}(x,y) \}$$

$$= \frac{1}{s^2} + \frac{\rho(s-1)^2}{s^2(s+1)^2}$$

The density function of $f_Z(z)$ is found by performing the inverse operation of $M_Z(s)$.

$$f_Z(z) = \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} \left\{ \frac{z^{-s}}{s^2} + \frac{\rho z^{-s}(s-1)^2}{s^2(s+1)^2} \right\} ds$$

The density of $f_Z(z)$ may be found by performing the inversion integral directly, or by summing the residues of two terms at $s = 0$ for term one and $s = 0$, $s = -1$ for term two.

For term one, R_1 = residue at $s = 0$.

$$R_1 = \frac{d}{ds} z^{-s} \bigg|_{s=0} = \ln(1/z) \quad 0 < z \leq 1$$

For term two, R_2 = residue at $s = 0$.

$$R_2 = \frac{d}{ds} \frac{\rho z^{-s}(s-1)^2}{(s+1)^2} \bigg|_{s=0} = -4\rho + \rho\ln(1/z) \quad 0 < z \leq 1$$

For term two, R_3 = residue at $s = -1$.

$$R_3 = \frac{d}{ds} \frac{\rho z^{-s}(s-1)^2}{s^2} \bigg|_{s=-1} = 4\rho z + 4\rho z\ln(1/z) \quad 0 < z \leq 1$$

Since all residues are valid for $0 < z \leq 1$, $f_Z(z)$ is equal to the sum of R_1, R_2, and R_3. Summing these terms and rearranging, $f_Z(z)$ is then given by

$$f_Z(z) = [\rho(4z + 1) + 1] \ln(1/z) + 4\rho(z - 1) \qquad 0 < z \leq 1$$

which is the same density function for Z which was derived in Example 1.3.

Theorem 2.5: If X_1, Y_1; X_2, Y_2; ...; X_n, Y_n are n pairwise independent continuous random variables, X_i, Y_i dependent for all i, with bivariate probability density functions $f_1(x_1, y_1)$, $f_2(x_2, y_2)$, ..., $f_n(x_n, y_n)$, $x_i, y_i > 0$ for $i = 1, 2, ..., n$, then the probability density function of the dependent random variables

$$Z = \prod_{i=1}^{n} X_i^{a_i} \quad ; \quad W = \prod_{i=1}^{n} Y_i^{b_i}$$

for a_i, b_i rational, $i=1, 2, ... n$, is given by

$$f_{Z,W}(z,w) = \begin{cases} M_2^{-1} \left[\prod_{i=1}^{n} M_{a_i s_1 - a_i + 1, b_i s_2 - b_i + 1} \{ f_i(x_i, y_i) \} \right] & z, w > 0 \\ \\ 0 & \text{otherwise} \end{cases} \qquad (2.27)$$

Proof of Theorem 2.5

The double Mellin transform of the probability density function $f_{Z,W}(z,w)$ is by definition

$$M_{s_1, s_2} \{ f_{Z,W}(z,w) \} = \int_0^\infty \int_0^\infty f_{Z,W}(z,w) z^{s_1 - 1} w^{s_2 - 1} \, dz \, dw$$

$$= E[z^{s_1 - 1} w^{s_2 - 1}]$$

where E is the expected value operator. From the definition of Z and

W, this becomes

$$M_{s_1,s_2}\{ f_{Z,W}(z,w) \} = E[\; \prod_{i=1}^{n} (x_i^{a_i})^{s_1-1} (y_i^{a_i})^{s_2-1} \;]$$

and, since X_i, X_j and Y_i, Y_j independent for all $i \neq j$,

$$M_{s_1,s_2}\{ f_{Z,W}(z,w) \} = \prod_{i=1}^{n} E[\; x_i^{a_i s_1 - a_i} y_i^{b_i s_2 - b_i} \;]$$

$$= \prod_{i=1}^{n} \int_0^\infty \int_0^\infty f_i(x_i,y_i) x_i^{a_i s_1 - a_i} y_i^{b_i s_2 - b_i} dx_i dy_i$$

$$= \prod_{i=1}^{n} M_{a_i s_1 - a_i + 1, b_i s_2 - b_i + 1}\{ f_i(x_i,y_i) \}$$

Applying the inverse Mellin transform, (2.27) follows.

Example 2.3:   Consider the Kellogg-Barnes I distribution given by

$$f_{X,Y}(x,y) = \frac{4\alpha^{\beta+1}}{\pi\Gamma(\beta+1)} (x^2 + y^2)^\beta e^{-\alpha(x^2 + y^2)} \qquad\qquad x,y > 0$$

Letting $r = x^2 + y^2$, the Mellin transform is given by

$$M(s_1,s_2) = \frac{4\alpha^{\beta+1}}{\pi\Gamma(\beta+1)} \int_0^\infty \int_0^{\pi/2} e^{-\alpha r^2} r^{s_1+s_2+2\beta-1} \cos\theta^{s_1-1} \sin\theta^{s_2-1} d\theta dr$$

$$= \frac{\alpha^{\beta+1}\Gamma(s_1/2)\Gamma(s_2/2)}{\pi\Gamma(\beta+1)\Gamma(s_1/2+s_2/2)} \int_0^\infty r^{s_1+s_2+2\beta-2} e^{-\alpha r^2} 2rdr$$

Letting $u = r^2$ and $du = 2rdr$, then

$$M(s_1,s_2) = \frac{\alpha^{\beta+1}\Gamma(s_1/2)\Gamma(s_2/2)}{\pi\Gamma(\beta+1)\Gamma(s_1/2+s_2/2)} \int_0^\infty u^{s_1/2+s_2/2+\beta-1} e^{-\alpha u} \, du$$

$$= \frac{\alpha\Gamma(s_1/2)\Gamma(s_2/2)\Gamma(\beta+s_1/2+s_2/2)}{\pi\Gamma(\beta+1)\Gamma(s_1/2+s_2/2)\alpha^{s_1/2+s_2/2}}$$

Now suppose the bivariate probability density function of $Z = X_1/X_2$, $W = Y_1/Y_2$ is desired where X_1,Y_1 and X_2,Y_2 are distributed according to the Kellogg-Barnes I distribution given above. Suppose further that $f_1(x_1,y_1)$ has parameters $\alpha_1 = 1$, $\beta_1 = 0$, and $f_2(x_2,y_2)$ has parameters $\alpha_2 = 1$, $\beta_2 = \beta$. Substituting the respective parameter values into the Mellin transform for the Kellogg-Barnes I distribution and from Theorem 2.5, $a_1 = b_1 = 1$, $a_2 = b_2 = -1$, the probability density function of Z, W is given by

$$f_{Z,W}(z,w) = M_2^{-1}\left[M_{s_1,s_2}\{ f_1(x_1,y_1) \} M_{2-s_1,2-s_2}\{ f_2(x_2,y_2) \} \right]$$

$$= M_2^{-1}\left[\frac{\Gamma(s_1/2)\Gamma(s_2/2)\Gamma(1-s_1/2)\Gamma(1-s_2/2)\Gamma(\beta+2-s_1/2-s_2/2)}{\pi^2\Gamma(\beta+1)\Gamma(2-s_1/2-s_2/2)} \right]$$

$$= \frac{\Gamma(\beta+2)}{\pi^2\Gamma(\beta+1)} M_2^{-1}\left[\frac{\Gamma(2)\Gamma(s_1/2)\Gamma(s_2/2)\Gamma(1-s_1/2)\Gamma(1-s_2/2)\Gamma(\beta+2-s_1/2-s_2/2)}{\Gamma(\beta+2)\Gamma(1)\Gamma(1)\Gamma(2-s_1/2-s_2/2)} \right]$$

Using property (2.9) for $a = b = 2$, then the density function for Z,W above can be rewritten as

$$f_{Z,W}(z^2,w^2) = \frac{4(\beta+1)}{\pi^2} M_2^{-1} \left[\frac{\Gamma(2)\Gamma(s_1)\Gamma(s_2)\Gamma(1-s_1)\Gamma(1-s_2)\Gamma(\beta+2-s_1-s_2)}{\Gamma(\beta+2)\Gamma(1)\Gamma(1)\Gamma(2-s_1-s_2)} \right]$$

Using Equation (2.15) for $b = b' = 1$, $a = \beta+2$, and $c = 2$, the inverse may be found directly and is given as

$$f_{Z,W}(z,w) = \frac{4(\beta+1)}{\pi^2} F_1(\beta+2;1,1;2;-z^2,-w^2)$$

where F_1 is Appell's hypergeometric function of two variables as defined in Appendix B. The series converges when $|x| < 1$ and $|y| < 1$. For other values of x and y, the function can be evaluated by the usual methods of analytic continuation.

Except for the normalization constant, the results given above are identical to the results derived by Fox (10). The distributions used by Fox were defined over the range $-\infty < x,y < \infty$.

By applying Theorem 2.5 followed by Theorem 2.4 or Equation (2.23), a general theorem for finding the distribution of a random variable which is the product, ratio, or power of an arbitrary number of bivariate random variables can be proved.

Theorem 2.6: If X_1,Y_1; X_2,Y_2; ...; X_n,Y_n are n pairwise independent continuous random variables, X_i,Y_i dependent for all i, with probability density functions $f_1(x_1,y_2),f_2(x_2,y_2)$, ..., $f_n(x_n,y_n)$, $x_i,y_i > 0$ i=1,2,...,n , then the probability density function of the random variable

$$Z = \prod_{i=1}^{n} X_i^{a_i} Y_i^{b_i}$$

for a_i, b_i rational, $i=1,2,\ldots,n$, is given by

$$f_Z(z) = \begin{cases} M_1^{-1} \left[\prod_{i=1}^{n} M_{a_i s - a_i + 1, b_i s - b_i + 1} \{ f_i(x_i, y_i) \} \right] & z > 0 \\ \\ 0 & \text{otherwise} \end{cases} \qquad (2.28)$$

The proof of Theorem 2.6 follows directly from application of (2.23) to Theorem 2.5.

The advantage of these theorems is that all the theorems developed by Carter (3) are special cases of these theorems. Therefore, these theorems provide the techniques for finding distributions of random variables which result from products, quotients, and powers of an arbitrary number of independent and dependent random variables. The only restriction is that any one random variable is correlated to at most only one other random variable.

CHAPTER 3

The H-function

3.1 General Remarks

The H-function was first introduced by Fox in 1961 as a
symmetric Fourier kernel to the G-function of Meijer and was used
extensively in physics and engineering. Carter (3) demonstrated the
importance of this function in statistical applications when viewed as
a probability distribution. The reasons for this importance are
two-fold. First, the H-function is the most general special function,
containing most of the other special functions as special cases.
Thus, anything accomplished with the general form for the H-function
is valid for all special cases. This allows the user to solve a
problem for a large class of functions with a single derivation.

The second advantage to H-functions is readily seen in the
following sections. The properties of the H-function are such that
they are reduced to simple adjustments of given parameters. The
simple parameter changes needed to find the Mellin transforms or the
derivatives of an H-function are trivial compared to performing these
same operations for various special cases. Indeed, the derivative of
an H-function is another H-function.

Carter (3) used these properties to show that products,
quotients, and rational powers of independent H-function variates
yield a random variable which also follows an H-function distribution.
These results provide a robust method for determining algebraic

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combinations of independent random variables and strong motivation for
extension of this theory to bivariate distributions.

A G-function of two variables was given by Sharma (34) in an
attempt to generalize classes of functions of two variables. Since
the H-function given by Fox is not a special case of this function,
several workers have extended the univariate H-function and called it
an H-function of two variables. The H-function of two variables
contains as special cases most of the known functions of one and two
variables, Appell's functions, G-function of two variables, Whittaker
functions of two variables, H-function of one variable, product of two
H-functions, etc.

Chapter 4 presents the bivariate H-function distribution, a
bivariate probability function, expressed in terms of an H-function
times an appropriate constant. Some of the classical bivariate
distributions are shown to be special cases of the bivariate
H-function distribution. Methods for solving resultant distributions
derived from products and quotients of dependent H-function variates
are presented. The important result is that such combinations result
in a distribution that is an H-function distribution of one variable.
Finally, methods for handling products and quotients of dependent
H-function variates from two bivariate H-function distributions which
are pairwise independent are given. Such combinations result in
distributions which are also bivariate H-function distributions.

3.2 Definitions.   (5:32;7:93;3:35;14:25;69:37;12)

Although there are slight variations in the definition of the H-function, this paper uses the H-function of one variable given by Cook (5:32) and a slight modification to the definition of the bivariate H-function given by Goyal (69:37). The H-function of one variable may be defined by

$$H(z) = H^{M,N}_{P,Q} [z : (\vartheta_j, \Theta_j) ; (\phi_j, \Phi_j)]$$

$$= \frac{1}{2\pi i} \int_C \frac{\prod\limits_{j=1}^{M} \Gamma(\phi_j + \Phi_j s) \prod\limits_{j=1}^{N} \Gamma(1 - \vartheta_j - \Theta_j s)}{\prod\limits_{j=N+1}^{P} \Gamma(\theta_j + \Theta_j s) \prod\limits_{j=M+1}^{Q} \Gamma(1 - \phi_j - \Phi_j s)} z^{-s} ds \qquad (3.1)$$

where C is a contour in the complex s-plane running from $w-i\infty$ to $w+i\infty$. The following assumptions are made.

   (i) M, N, P, Q are integers such that $0 \le M \le Q$ and $0 \le N \le P$

   (ii) parameters ϑ_j, ϕ_j are complex numbers and Θ_j, Φ_j are positive real numbers

   (iii) empty products are defined to be equal to unity

   (iv) all poles of $\Gamma(\phi_j + \Phi_j s)$ lie to the left of C, and all poles of $\Gamma(1 - \vartheta_j - \Theta_j s)$ lie to the right

The bivariate H-function is defined by

$$H[x,y] = H^{M_1,N_1,M_2,N_2,M_3,N_3}_{P_1,Q_1,P_2,Q_2,P_3,Q_3} \left[x \middle| y \middle| \begin{array}{c} (e_i,E_i) \\ (a_i,A_i) \; ; \; (c_i,C_i) \\ (f_i,F_i) \\ (b_i,B_i) \; ; \; (d_i,D_i) \end{array} \right]$$

$$= \frac{1}{(2\pi i)^2} \int_{C_1} \int_{C_2} x_1(-s_1) x_2(-s_2) x_3(-s_1-s_2) x^{s_1} y^{s_2} \, ds_1 ds_2 \quad (3.2)$$

where an empty product is denoted by unity, C_1 is a contour in the complex s_1 plane running from $h-i\infty$ to $h+i\infty$, C_2 is a contour in the complex s_2 plane running from $w-i\infty$ to $w+i\infty$, and

$$x_1(-s_1) = \frac{\prod\limits_{1}^{M_1} \Gamma(b_i - B_i s_1) \; \prod\limits_{1}^{N_1} \Gamma(1-a_i+A_i s_1)}{\prod\limits_{N_1+1}^{P_1} \Gamma(a_i - A_i s_1) \; \prod\limits_{M_1+1}^{Q_1} \Gamma(1-b_i+B_i s_1)} \quad (3.3)$$

$$x_2(-s_2) = \frac{\prod\limits_{1}^{M_2} \Gamma(d_i - D_i s_2) \; \prod\limits_{1}^{N_2} \Gamma(1-c_i+C_i s_2)}{\prod\limits_{N_2+1}^{P_2} \Gamma(c_i - C_i s_2) \; \prod\limits_{M_2+1}^{Q_2} \Gamma(1-d_i+D_i s_2)} \quad (3.4)$$

$$x_3(-s_1-s_2) = \frac{\prod\limits_{1}^{M_3} \Gamma(e_i - E_i(s_1+s_2)) \; \prod\limits_{1}^{N_3} \Gamma(1-f_i+F_i(s_1+s_2))}{\prod\limits_{N_3+1}^{P_3} \Gamma(f_i - F_i(s_1+s_2)) \; \prod\limits_{M_3+1}^{Q_3} \Gamma(1-e_i+E_i(s_1+s_2))} \quad (3.5)$$

The following assumptions are made.

(i) M_i, N_i, P_i, Q_i, $i=1,2,3$ are non-negative integers such that $0 \leq N_i \leq P_i$, $0 \leq M_i \leq Q_i$, $i=1,2,3$

(ii) parameters a_i, b_i, c_i, d_i, e_i, f_i are real or complex and parameters A_i, B_i, C_i, D_i, E_i, F_i are real positive numbers

(iiia) poles of $\Gamma(b_i-B_i s_1)$, $(i=1\ldots M_1)$, $\Gamma(e_i-E_i(s_1+s_2))$, $(i=1\ldots M_3)$ lie to the right of C_1 and poles of $\Gamma(1-a_i+A_i s_1)$, $(i=1\ldots N_1)$, $\Gamma(1-f_i+F_i(s_1+s_2)$, $(i=1\ldots N_3)$ lie to the left

(iva) poles of $\Gamma(d_i-D_i s_2)$, $(i=1\ldots M_2)$, $\Gamma(e_i-E_i(s_1+s_2)$, $(i=1\ldots M_3)$ lie to the right of C_2 and poles of $\Gamma(1-c_i+C_i s_2)$, $(i=1\ldots N_2)$, $\Gamma(1-f_i+F_i(s_1+s_2))$, $(i=1\ldots N_3)$ lie to the left.

If $-s_1$ is substituted for s_1 and $-s_2$ for s_2 and recognizing that

$$\int_b^a f(-s)d(-s) = \int_a^b f(-s)ds$$

and perform that operation twice, then $H[x,y]$ may be redefined as

$$H[x,y] = \frac{1}{(2\pi i)^2} \int_{C_1} \int_{C_2} x_1(s_1)x_2(s_2)x_3(s_1+s_2)x^{-s_1} y^{-s_2} ds_1 ds_2 \qquad (3.6)$$

where

$$x(s_1) = \frac{\prod\limits_{1}^{M_1} \Gamma(b_i+B_i s_1) \quad \prod\limits_{1}^{N_1} \Gamma(1-a_i-A_i s_1)}{\prod\limits_{N_1+1}^{P_1} \Gamma(a_i+A_i s_1) \quad \prod\limits_{M_1+1}^{Q_1} \Gamma(1-b_i-B_i s_1)} \tag{3.7}$$

$$x_2(s_2) = \frac{\prod\limits_{1}^{M_2} \Gamma(d_i+D_i s_2) \quad \prod\limits_{1}^{N_2} \Gamma(1-c_i-C_i s_2)}{\prod\limits_{N_2+1}^{P_2} \Gamma(c_i+C_i s_2) \quad \prod\limits_{M_2+1}^{Q_2} \Gamma(1-d_i-D_i s_2)} \tag{3.8}$$

$$x_3(s_1+s_2) = \frac{\prod\limits_{1}^{M_3} \Gamma(e_i+E_i(s_1+s_2)) \quad \prod\limits_{1}^{N_3} \Gamma(1-f_i-F_i(s_1+s_2))}{\prod\limits_{N_3+1}^{P_3} \Gamma(f_i+F_i(s_1+s_2)) \quad \prod\limits_{M_3+1}^{Q_3} \Gamma(1-e_i-E_i(s_1+s_2))} \tag{3.9}$$

Assumptions (i) and (ii) remain unchanged and assumptions (iiia) and (iva) are changed to

(iiib) poles of $\Gamma(b_i+B_i s_1)$, $(i=1\ldots M_1)$, $\Gamma(e_i+E_i(s_1+s_2))$, $(i=1\ldots M_3)$ lie to the left of C_1 and poles of $\Gamma(1-a_i-A_i s_1)$, $(i=1\ldots N_1)$, $\Gamma(1-f_i-F_i(s_1+s_2))$, $(i=1\ldots N_3)$ lie to the right.

(ivb) poles of $\Gamma(d_i+D_i s_2)$, $(i=1\ldots M_2)$, $\Gamma(e_i+E_i(s_1+s_2))$, $(i=1\ldots M_3)$ lie to the left of C_2 and poles of $\Gamma(1-c_i-C_i s_2)$, $(i=1\ldots N_2)$, $\Gamma(1-f_i-F_i(s_1+s_2))$, $(i=1\ldots N_3)$ lie to the right.

This second form of the definition has an advantage in that it is of the form of a double Mellin integral transform inversion integral. Form (3.6) of the H-function definition is used hereafter,

because of the direct relation for the Mellin transform.

Form (3.6) of the H-function definition is useful in that its Mellin transform is $x_1(s_1)x_2(s_2)x_3(s_1+s_2)$. It is clear from this definition that the bivariate H-function can represent functions whose Mellin transforms have terms of the form $x_3(s_1+s_2)$ or $x_3(-s_1-s_2)$. Practically speaking, it is very likely that a great many functions will have Mellin transforms with terms of the form $x_3(s_1-s_2)$ or $x_3(s_2-s_1)$. It is possible to modify definition (3.6) to accomodate such functions.

Using definition (3.6), if $-s_1$ is substituted for s_1, then an equivalent representation for (3.6) is

$$H[x,y] = \frac{1}{(2\pi i)^2} \int_{C_1} \int_{C_2} x_1(-s_1)x_2(s_2)x_3(-s_1+s_2)x^{s_1} y^{-s_2} ds_1 ds_2 \quad (3.10)$$

Similarly, if $-s_2$ is substituted for s_2 in (3.6)

$$H[x,y] = \frac{1}{(2\pi i)^2} \int_{C_1} \int_{C_2} x_1(s_1)x_2(-s_2)x_3(s_1-s_2)x^{-s_1} y^{s_2} ds_1 ds_2 \quad (3.11)$$

where the poles of the x_2 and the x_3 terms are interchanged about the C_2 axis in the s_2 plane for (3.11) and the poles of the x_1 and the x_3 terms are interchanged about the C_1 axis in the s_1 plane for (3.10).

From this it is clear that the mathematics of the H-function can handle functions whose Mellin transforms have terms of the form $x_3(s_1-s_2)$ or $x_3(s_2-s_1)$. Definition (3.6) can now be modified to accomodate such functions.

Definition: A bivariate functional whose Mellin transform is given by $x_1(s_1)x_2(-s_2)x_3(s_1-s_2)$ has the bivariate H-function given by

$$H[x,y] = {}_2H^{M_1,N_1,M_2,N_2,M_3,N_3}_{P_1,Q_1,P_2,Q_2,P_3,Q_3} \left[\begin{array}{c|c} & (e_i,E_i) \\ x & (a_i,A_i) \; ; \; (c_i,C_i) \\ y & (f_i,F_i) \\ & (b_i,B_i) \; ; \; (d_i,D_i) \end{array} \right]$$

$$= \frac{1}{(2\pi i)^2} \int_{C_1} \int_{C_2} x_1(s_1)x_2(s_2)x_3(s_1+s_2)x^{-s_1} y^{s_2} ds_1 ds_2 \qquad (3.12)$$

where $x_1(s_1)$, $x_2(s_2)$, and $x_3(s_1+s_2)$ are defined by (3.7), (3.8), and (3.9). Assumptions (i), (ii), (iiib), and (ivb) hold.

Henceforth, $_1H[x,y]$ shall be used to denote definition (3.6) and $_2H[x,y]$ to denote definition (3.12). The relationship between $_1H[x,y]$ and $_2H[x,y]$ is readily derived by observing that y^{s_2} may be rewritten as $(1/y)^{-s_2}$ in (3.12). Comparing (3.12) with this substitution to (3.6), it is readily seen that the following relationship holds.

$$_2H[x,y] = {}_1H[x,1/y] \qquad (3.13)$$

3.3 Properties: (14:24-25;69:39-41;12)

Making a change of variable substitution in (3.2) yields the following identities which are useful in bivariate manipulations:

$$H[x,y] = {}_rH^{M_1,N_1,M_2,N_2,M_3,N_3}_{P_1,Q_1,P_2,Q_2,P_3,Q_3} \left[\begin{array}{c|cc} & & (e_i,E_i) \\ 1/x & (a_i,A_i) & ; & (c_i,C_i) \\ 1/y & & (f_i,F_i) \\ & (b_i,B_i) & ; & (d_i,D_i) \end{array} \right]$$

$$= {}_rH^{N_1,M_1,N_2,M_2,N_3,M_3}_{Q_1,P_1,Q_2,P_2,Q_3,P_3} \left[\begin{array}{c|cc} & & (1-f_i,F_i) \\ x & (1-b_i,B_i) & ; & (1-d_i,D_i) \\ y & & (1-e_i,E_i) \\ & (1-a_i,A_i) & ; & (1-c_i,C_i) \end{array} \right] \qquad (3.14)$$

For $k > 0$, the following is true.

$$k^2 {}_rH^{M_1,N_1,M_2,N_2,M_3,N_3}_{P_1,Q_1,P_2,Q_2,P_3,Q_3} \left[\begin{array}{c|cc} & & (e_i,kE_i) \\ x^k & (a_i,kA_i) & ; & (c_i,kC_i) \\ y^k & & (f_i,kF_i) \\ & (b_i,kB_i) & ; & (d_i,kD_i) \end{array} \right]$$

$$= {}_rH^{M_1,N_1,M_2,N_2,M_3,N_3}_{P_1,Q_1,P_2,Q_2,P_3,Q_3} \left[\begin{array}{c|cc} & & (e_i,E_i) \\ x & (a_i,A_i) & ; & (c_i,C_i) \\ y & & (f_i,F_i) \\ & (b_i,B_i) & ; & (d_i,D_i) \end{array} \right] \qquad (3.15)$$

Making use of the Mellin transform property $M_{s_1,s_2}\{ x^m y^n f_{X,Y}(x,y) \} = M_{s_1+m,s_2+n}\{ f_{X,Y}(x,y) \}$, the following property is obtained.

$$x^m y^n \,_1H^{M_1,N_1,M_2,N_2,M_3,N_3}_{P_1,Q_1,P_2,Q_2,P_3,Q_3} \left[x \atop y \middle| \begin{array}{c} (e_i,E_i) \\ (a_i,A_i) \; ; \; (c_i,C_i) \\ (f_i,F_i) \\ (b_i,B_i) \; ; \; (d_i,D_i) \end{array} \right]$$

$$= \,_1H^{M_1,N_1,M_2,N_2,M_3,N_3}_{P_1,Q_1,P_2,Q_2,P_3,Q_3} \left[x \atop y \middle| \begin{array}{c} (e_i+E_i(m+n),E_i) \\ (a_i+A_i m,A_i) \; ; \; (c_i+C_i n,C_i) \\ (f_i+F_i(m+n),F_i) \\ (b_i+B_i m,B_i) \; ; \; (d_i+D_i n,D_i) \end{array} \right] \qquad (3.16)$$

$$x^m y^n \,_2H^{M_1,N_1,M_2,N_2,M_3,N_3}_{P_1,Q_1,P_2,Q_2,P_3,Q_3} \left[x \atop y \middle| \begin{array}{c} (e_i,E_i) \\ (a_i,A_i) \; ; \; (c_i,C_i) \\ (f_i,F_i) \\ (b_i,B_i) \; ; \; (d_i,D_i) \end{array} \right]$$

$$= \,_2H^{M_1,N_1,M_2,N_2,M_3,N_3}_{P_1,Q_1,P_2,Q_2,P_3,Q_3} \left[x \atop y \middle| \begin{array}{c} (e_i+E_i(m-n),E_i) \\ (a_i+A_i m,A_i) \; ; \; (c_i-C_i n,C_i) \\ (f_i+F_i(m-n),F_i) \\ (b_i+B_i m,B_i) \; ; \; (d_i-D_i n,D_i) \end{array} \right] \qquad (3.17)$$

3.4 Mellin Transform: (3:37;5:35;7:102;12)

Definition (3.1) of the H-function is exactly that of a Mellin transform inversion integral so that the Mellin transform of the H-function is directly given as

$$
M_s\{H(cz)\} = c^{-s} \frac{\prod\limits_{i=1}^{M} \Gamma(\phi_i + \Phi_i s) \prod\limits_{i=1}^{N} \Gamma(1 - \theta_i - \Theta_i s)}{\prod\limits_{i=N+1}^{P} \Gamma(\vartheta_i + \Theta_i s) \prod\limits_{i=M+1}^{Q} \Gamma(1 - \phi_i - \Phi_i s)} \tag{3.18}
$$

Form (3.6) of the bivariate H-function definition is exactly that of a double Mellin transform inversion integral, so that the Mellin integral transform of $_1H[x,y]$ is directly given as

$$
M_{s_1,s_2}\{ _1H[g_1 x, g_2 y] \} = x_1(s_1) x_2(s_2) x_3(s_1 + s_2) g_1^{-s_1} g_2^{-s_2} \tag{3.19}
$$

where $x_1(s_1)$, $x_2(s_2)$, and $x_3(s_1 + s_2)$ are defined by equations (3.7), (3.8), and (3.9) respectively.

Under definition (3.12) for $_2H[x,y]$ and assuming convergence of the integral in the definition, the Mellin transform can be found by interpreting the bivariate H-function as the inverse Mellin transform of the coefficients on $x^{-s_1} y^{-s_2}$. Then

$$
_2H[g_1 x, g_2 y] = \frac{1}{(2\pi i)^2} \int_{C_1} \int_{C_2} x_1(s_1) x_2(-s_2) x_3(s_1 - s_2)(g_1 x)^{-s_1} (g_2 y)^{-s_2} ds_1 ds_2
$$

where

$$x_3(s_1-s_2) = \frac{\overset{M_3}{\underset{1}{\Pi}}\,\Gamma(e_i+E_i(s_1-s_2))\;\;\overset{N_3}{\underset{1}{\Pi}}\,\Gamma(1-f_i-F_i(s_1-s_2))}{\underset{N_3+1}{\overset{P_3}{\Pi}}\,\Gamma(f_i+F_i(s_1-s_2))\;\;\underset{M_3+1}{\overset{Q_3}{\Pi}}\,\Gamma(1-e_i-E_i(s_1-s_2))} \tag{3.20}$$

Using the definition of the Mellin transform, one can express $_2H[g_1x,g_2y]$ in the form

$$_2H[g_1x,g_2y] = M_2^{-1}[\; x_1(s_1)x_2(-s_2)x_3(s_1-s_2)g_1^{-s_1}g_2^{-s_2}\;]$$

where M_2^{-1} is the inverse operation for the double Mellin transform as defined by Fox (10). It follows that

$$M_{s_1,s_2}\{_2H[g_1x,g_2y]\} = x_1(s_1)x_2(-s_2)x_3(s_1-s_2)g_1^{-s_1}g_2^{-s_2} \tag{3.21}$$

3.5 **Special Cases:** (14:26-28;16:569-571;12)

From the results of Reed (16) on double Mellin transforms of Appells functions, the following special cases for the H-function of two variables may be derived.

$$_1H\,{}^{1,1,1,1,0,1}_{1,1,1,1,1,1}\left[\begin{matrix} x \\ y \end{matrix}\,\middle|\,\begin{matrix} (1-c,1) \\ (1-b,1)\;;\;(1-b',1) \\ (1-a,1) \\ (0,1)\;;\;(0,1) \end{matrix}\right]$$

$$= \frac{\Gamma(a)\Gamma(b)\Gamma(b')}{\Gamma(c)}F_1(a,b,b';c;-x,-y) \tag{3.22}$$

$$
{}_1H \begin{array}{l} 1,1,1,1,0,1 \\ \\ 1,2,1,2,1,0 \end{array} \left[\begin{array}{l} x \\ \\ y \end{array} \left| \begin{array}{c} ----- \\ (1-b,1) \ ; \ (1-b',1) \\ (1-a,1) \\ (0,1),(1-c,1) \ ; \ (0,1),(1-c',1) \end{array} \right. \right]
$$

$$
= \frac{\Gamma(a)\Gamma(b)\Gamma(b')}{\Gamma(c)\Gamma(c')} \ F_2(a,b,b';c,c';-x,-y) \tag{3.23}
$$

$$
{}_1H \begin{array}{l} 1,2,1,2,0,0 \\ \\ 2,1,2,1,0,1 \end{array} \left[\begin{array}{l} x \\ \\ y \end{array} \left| \begin{array}{c} (1-c,1) \\ (1-a,1),(1-b,1) \ ; \ (1-a',1),(1-b',1) \\ ----- \\ (0,1) \ ; \ (0,1) \end{array} \right. \right]
$$

$$
= \frac{\Gamma(a)\Gamma(a')\Gamma(b)\Gamma(b')}{\Gamma(c)} \ F_3(a,a',b,b';c;-x,-y) \tag{3.24}
$$

$$
{}_1H \begin{array}{l} 1,0,1,0,0,2 \\ \\ 0,2,0,2,2,0 \end{array} \left[\begin{array}{l} x \\ \\ y \end{array} \left| \begin{array}{c} ----- \\ ---- \ ; \ ---- \\ (1-a,1),(1-b,1) \\ (0,1),(1-c,1) \ ; \ (0,1),(1-c',1) \end{array} \right. \right]
$$

$$
= \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(c')} \ F_4(a,b;c,c';-x,-y) \tag{3.25}
$$

In addition to the special cases listed above, Mathai and Saxena (14:26) provide an H-function of two variables identity for

Kampé de Feriét's function.

$$
{}_1H^{1,B,1,B,0,A}_{B,D+1,B,D+1,A,C}\left[\begin{array}{c|c}
x & (1-c_i,1) \\
& (1-b_i,1) \;;\; (1-b_i{}',1) \\
y & (1-a_i,1) \\
& (0,1),(1-d_i,1) \;;\; (0,1),(1-d_i{}',1)
\end{array}\right]
$$

$$
= \frac{\prod\limits_1^A \Gamma(a_i) \; \prod\limits_1^B \Gamma(b_i)\Gamma(b_i{}')}{\prod\limits_1^C \Gamma(c_i) \; \prod\limits_1^D \Gamma(d_i)\Gamma(d_i{}')} \; F\left[\begin{array}{c|cc}
A & a_1;\ldots a_A & \\
B & b_1,b_1{}';\ldots b_B,b_B{}' & -x,-y \\
C & c_1;\ldots c_C & \\
D & d_1,d_1{}';\ldots d_D,d_D{}' &
\end{array}\right] \quad (3.26)
$$

The identity given by Mathai and Saxena is of a slightly different form due to the fact that they used the definition of the form of (3.2). The identity above results from the definition for the H-function of two variables given by (3.6).

If $M_3 = N_3 = P_3 = Q_3 = 0$, the H-function of two variables breaks up into a product of two H-functions.

$$
{}_1H^{M_1,N_1,M_2,N_2,0,0}_{P_1,Q_1,P_2,Q_2,0,0}\left[\begin{array}{c|c}
x & ----- \\
& (a_i,A_i) \;;\; (c_i,C_i) \\
y & ----- \\
& (b_i,B_i) \;;\; (d_i,D_i)
\end{array}\right]
$$

$$= H^{M_1,N_1}_{P_1,Q_1} \left[x \, \middle| \, \begin{array}{c} (a_i, A_i) \\ (b_i, B_i) \end{array} \right] H^{M_2,N_2}_{P_2,Q_2} \left[y \, \middle| \, \begin{array}{c} (c_i, C_i) \\ (d_i, D_i) \end{array} \right] \qquad (3.27)$$

$$= H_1(x) \cdot H_2(y)$$

Here, $H_1(x)$ and $H_2(y)$ are univariate H-functions as defined by (3.1). Similar results hold for $_2H[x,y]$.

$$_2H^{M_1,N_1,M_2,N_2,0,0}_{P_1,Q_1,P_2,Q_2,0,0} \left[\begin{array}{c} x \\ \\ y \end{array} \, \middle| \, \begin{array}{c} ----- \\ (a_i, A_i) \; ; \; (c_i, C_i) \\ ----- \\ (b_i, B_i) \; ; \; (d_i, D_i) \end{array} \right]$$

$$= H^{M_1,N_1}_{P_1,Q_1} \left[x \, \middle| \, \begin{array}{c} (a_i, A_i) \\ (b_i, B_i) \end{array} \right] \cdot H^{N_2,M_2}_{Q_2,P_2} \left[y \, \middle| \, \begin{array}{c} (1-d_i, D_i) \\ (1-c_i, C_i) \end{array} \right] \qquad (3.28)$$

From (3.27) and (3.28), it is clear that for $M_i = N_i = P_i = Q_i = 0$, $i = 2,3$, the bivariate H-function reduces to a univariate H-function as defined by (3.1). It then stands to reason that distributional analysis of products and quotients of H-function variates in the univariate domain is a special case of distributional analysis of H-functions of higher order. This fact shall be demonstrated in Chapter 4.

Example 3.1: Consider once again Morgensterm's bivariate uniform distribution. The double Mellin transform was given in Example 2.2 as

$$M_{X,Y}(s_1,s_2) = \frac{1}{s_1 s_2} + \frac{\rho(s_1 - 1)(s_2 - 1)}{s_1 s_2 (s_1 + 1)(s_2 + 1)}$$

Noting that $(s + n) = \dfrac{\Gamma(s + n + 1)}{\Gamma(s + n)}$

$M_{X,Y}(s_1,s_2)$ may be rewritten in terms of gamma functions.

$$M_{X,Y}(s_1,s_2) = \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+1)\Gamma(s_2+1)} + \frac{\rho\Gamma(s_1)\Gamma(s_2)\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+2)\Gamma(s_2+2)\Gamma(s_1-1)\Gamma(s_2-1)}$$

While this form is not immediately expressible as a single H-function of two variables, it may be redefined as a sum of two H-functions of two variables.

$$f_{X,Y}(x,y) = H_1[\ x,y\ ] + H_2[\ x,y\ ]$$

where

$$H_1[\ x,y\ ] = {}_1H^{1,0,1,0,0,0}_{1,1,1,1,0,0} \left[\begin{array}{c|c} & \text{-----} \\ x & (1,1)\ ;\ (1,1) \\ y & \text{-----} \\ & (0,1)\ ;\ (0,1) \end{array} \right]$$

and

$$H_2[\ x,y\ ] = \rho\,{}_1H^{2,0,2,0,0,0}_{2,2,2,2,0,0} \left[\begin{array}{c|c} & \text{-----} \\ x & (2,1),(-1,1)\ ;\ (2,1),(-1,1) \\ y & \text{-----} \\ & (0,1),(0,1)\ ;\ (0,1),(0,1) \end{array} \right]$$

In this instance, it must be remembered that $H_1[\ x,y\ ]$ and $H_2[\ x,y\ ]$ are not H-functional representations of density functions, but are two general H-functions of two variables, the sum of which is a bivariate density representation.

While it is desirable to formulate $f_{X,Y}(x,y)$ as a single H-function of two variables, the representation above is still valuable in that given $Z = XY$ or $Z = X/Y$, $f_Z(z)$ may be represented as the sum of two H-functions of one variable which can be derived from $H_1[\ x,y\ ]$ and $H_2[\ x,y\ ]$. Numerical inversion of $H_1[\ z\ ]$ and $H_2[\ z\ ]$ can be accomplished by methods presented by Eldred and Cook. Numerical evaluation of $f_Z(z)$ can then be derived by appropriately summing the inversions of $H_1[\ z\ ]$ and $H_2[\ z\ ]$.

CHAPTER 4

The H-function Distribution

4.1 General Remarks

In this chapter a new bivariate probability density function
based on the H-function of two variables is introduced. The new
distribution, called the bivariate H-function distribution, includes
as special cases many of the more common bivariate distributions - the
bivariate gamma, the bivariate beta, and the bivariate Cauchy. Three
new bivariate distributions, called the Kellogg-Barnes distributions,
are also shown to be special cases of the bivariate H-function
distribution. Also, by extension, all the univariate H-function
distributions are special cases.

Formulas for finding the moments of the bivariate H-function
distribution and the normalizing constant are given. The cumulative
of the bivariate H-function distribution is shown to be another
H-function, a property that is not shared by other bivariate
distributions. Examples are given.

4.2 Definitions:

Definition: Consider a random variable Z with probability
density function given by

$$f_Z(z) = \begin{cases} k\ H(cz), & cz \in S \\ 0, & \text{otherwise} \end{cases} \qquad (4.1)$$

where $H(cz)$ represents a univariate H-function as defined in section
3.2, k and c are real constants such that

$$\int_0^\infty f_Z(z) \, dz = 1$$

and S is a subset of the positive real values u for which H(u) is convergent. The random variable Z will then be called an H-function variate or a random variable with an H-function distribution (3:41;7:103;5:84;17:200).

Definition: Consider the random variables X, Y with joint probability density function given by

$$f_{X,Y}(x,y) = \begin{cases} k_r H[g_1 x, g_2 y] \, , & g_1 x \in S_1 \, , \; g_2 y \in S_2 \\ 0 \, , & \text{otherwise} \end{cases} \quad (4.2)$$

where $_r H[g_1 x, g_2]$ represents an H-function as defined in section 3.2, k, g_1, and g_2 are real constants such that

$$\int_0^\infty \int_0^\infty f_{X,Y}(x,y) \, dxdy = 1 \, ,$$

and S_1, S_2 are subsets of the positive real values u,v for which H(u,v) is convergent. The random variables X, Y will then be called dependent H-function variates or random variables with a bivariate H-function distribution.

4.3 Special Cases

Since the univariate H-function distribution may be expressed as a special case of (4.2), all of the classical univariate non-negative probability distributions studied by Carter (1972), Eldred (1979), and Cook (1981) are special cases of the bivariate

H-function distribution. In addition, some of the classical non-negative bivariate distributions may be expressed in the form (4.2). Converting a probability density function into its H-function form is accomplished by taking the Mellin transform of the density function and arranging the transform such that it is products and quotients of gamma functions. The H-function form may then be identified by taking the inverse Mellin transform.

(i) McKay's bivariate gamma distribution

$$f_{X,Y}(x,y) = \frac{a^{p+q}}{\Gamma(p)\Gamma(q)} \, x^{p-1} \, (y-x)^{q-1} \, e^{-ay} \qquad \begin{array}{l} y>x>0 \\ a,p,q>0 \end{array}$$

Let $c = (a^{p+q})/(\Gamma(p)\Gamma(q))$. Taking the Mellin transform and integrating first with respect to x and then with respect to y yields

$$M(s_1,s_2) = c\int_0^\infty \int_0^y x^{p-1} \, (y-x)^{q-1} \, e^{-ay} \, x^{s_1-1} \, y^{s_2-1} \, dxdy$$

$$= c\int_0^\infty y^{s_2-1} \, e^{-ay} \, [\int_0^y x^{s_1+p-2} \, (y-x)^{q-1} \, dx] \, dy$$

Using the Mellin transform property (2.8), the transform above becomes

$$M(s_1,s_2) = c\int_0^\infty y^{s_2-1} e^{-ay} \, [\int_0^y x^{z-1} \, (y-x)^{q-1} \, dx] \Big|_{z=s_1+p-1} dy$$

Using (15:16, # 2.20) to evaluate the inner integral, the transform can now be written

$$M(s_1,s_2) = \frac{c\Gamma(q)\Gamma(s_1+p-1)}{\Gamma(s_1+p+q-1)} \int_0^\infty y^{s_1+s_2+p+q-3} e^{-ay} dy$$

Realizing that the integral is just the gamma function and replacing the value for c yields

$$M(s_1,s_2) = \frac{a^{2-s_1-s_2} \Gamma(p-1+s_1)\Gamma(p+q-2+s_1+s_2)}{\Gamma(p)\Gamma(p+q-1+s_1)}$$

$f_{X,Y}(x,y)$ is returned by taking the inverse Mellin transform given by (2.6).

$$f_{X,Y}(x,y) = \frac{a^2}{\Gamma(p)} \frac{1}{(2\pi i)^2} \iint \frac{\Gamma(p-1+s_1)\Gamma(p+q-2+s_1+s_2)}{\Gamma(p+q-1+s_1)} (ax)^{-s_1}(ay)^{-s_2} ds_1 ds_2$$

$$= \frac{a^2}{\Gamma(p)} {_1H_{1,1,0,0,0,1}^{1,0,0,0,1,0}} \left[\begin{array}{c|c} ax \\ ay \end{array} \begin{array}{c} (p+q-2,1) \\ (p+q-1,1) \; ; \; ----- \\ ----- \\ (p-1,1) \; ; \; ----- \end{array} \right] \tag{4.3}$$

(ii) **The bivariate beta distribution**

$$f_{X,Y}(x,y) = \frac{\Gamma(p_1+p_2+p_3)}{\Gamma(p_1)\Gamma(p_2)\Gamma(p_3)} x^{p_1-1} y^{p_2-1} (1-x-y)^{p_3-1}$$

$$x,y>0 \; , \; x+y\leq 1 \; , \; p_1,p_2,p_3>0.$$

Let $c = \Gamma(p_1+p_2+p_3)/\Gamma(p_1)\Gamma(p_2)\Gamma(p_3)$. Taking the Mellin transform and integrating first with respect to y and then with respect to x yields

$$M(s_1,s_2) = c\int_0^1 \int_0^{1-x} x^{p_1-1} \, y^{p_2-1} \, (1-x-y)^{p_3-1} \, x^{s_1-1} \, y^{s_2-1} \, dydx$$

$$= c\int_0^1 x^{(s_1+p_1-1)-1} \, [\int_0^{1-x} y^{(s_2+p_2-1)-1} (1-x-y)^{p_3-1} dy] \, dx$$

Using (15:16 # 2.20) to evaluate the inner integral and substituting the value of c, the transform can now be written

$$M(s_1,s_2) = \frac{\Gamma(p_1+p_2+p_3)\Gamma(s_2+p_2-1)}{\Gamma(p_1)\Gamma(p_2)\Gamma(s_2+p_2+p_3-1)} \int_0^1 x^{(s_1+p_1-1)-1}(1-x)^{s_2+p_2+p_3-2} dx$$

Using (15:16 # 2.20) once again to evaluate the integral yields

$$M(s_1,s_2) = \frac{\Gamma(p_1+p_2+p_3)\Gamma(s_1+p_1-1)\Gamma(s_2+p_2-1)}{\Gamma(p_1)\Gamma(p_2)\Gamma(s_1+s_2+p_1+p_2+p_3-2)}$$

The H-function form is found by using the Mellin transform identity for the H-function as defined by (3.18) or (3.20).

$$f_{X,Y}(x,y) = M_2^{-1}[M(s_1,s_2)]$$

$$= \frac{\Gamma(p_1+p_2+p_3)}{\Gamma(p_1)\Gamma(p_2)} \, {}_1H^{1,0,1,0,0,0}_{0,1,0,1,1,0} \left[\begin{array}{c|c} x & \begin{array}{c} ----- \\ ----- \; ; \; ----- \\ (p_1+p_2+p_3-2,1) \\ (p_1-1,1) \; ; \; (p_2-1,1) \end{array} \\ y & \end{array} \right] \quad (4.4)$$

(iii) The quarter Cauchy distribution

$$f_{X,Y}(x,y) = \frac{2c}{\pi} (c^2 + x^2 + y^2)^{-3/2} , \qquad x,y>0 , c>0$$

Letting $r^2 = x^2 + y^2$, $x = \cos\theta$, $y = \sin\theta$, and $dxdy = rdrd\theta$, the Mellin transform can be written as

$$M(s_1,s_2) = \frac{2c}{\pi} \int_0^\infty \int_0^{\pi/2} (r\cos\theta)^{s_1-1} (r\sin\theta)^{s_2-1} (c^2+r^2)^{-3/2} rd\theta dr$$

$$= \frac{2c}{\pi} \int_0^\infty r^{s_1+s_2-1} (c^2+r^2)^{-3/2} [\int_0^{\pi/2} \cos\theta^{s_1-1} \sin\theta^{s_2-1} d\theta] dr$$

$$= \frac{c}{\pi} B(s_1/2,s_2/2) \int_0^\infty r^{s_1+s_2-1} (c^2+r^2)^{-3/2} dr$$

where $B(u,v)$ is the beta function and is given by

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$$

Using (15:15 # 2.19) to evaluate the integral and Mellin transform property (2.8), the transform becomes

$$M(s_1,s_2) = \frac{c\Gamma(s_1/2)\Gamma(s_2/2)}{2\pi\Gamma(s_1+s_2)} \frac{c^{s_1+s_2}\Gamma(s_1/2+s_2/2)\Gamma(3/2-s_1/2-s_2/2)}{c^3\Gamma(3/2)}$$

$$= \frac{1}{c^2 \pi^{3/2}} \; c^{s_1+s_2} \; \Gamma(s_1/2)\Gamma(s_2/2)\Gamma(3/2-s_1/2-s_2/2)$$

Using the Mellin transform identity (3.18) for the H-function, the H-function form for $f_{X,Y}(x,y)$ can be found.

$$f_{X,Y}(x,y) = M_2^{-1}[\; M(s_1,s_2)\;]$$

$$= \frac{1}{c^2 \pi^{3/2}} \; {}_1H^{1,0,1,0,0,1}_{0,1,0,1,1,0} \left[\begin{array}{c|c} \frac{1}{c} x & \begin{array}{c} \text{-----} \\ \text{-----} \; ; \; \text{-----} \\ (-1/2,1/2) \end{array} \\ \frac{1}{c} y & (0,1/2) \; ; \; (0,1/2) \end{array} \right] \qquad (4.5)$$

(iv) **Kellogg-Barnes Type I distribution**

$$f_{X,Y}(x,y) = \frac{4\alpha^{\beta+1}}{\pi\Gamma(\beta+1)} (x^2 + y^2)^\beta \; e^{-\alpha(x^2 + y^2)} \; , \qquad \begin{array}{l} x,y>0 \\ \alpha,\beta>0 \end{array}$$

The Mellin transform for the Kellogg-Barnes I distribution was computed in Example 2.3 and is given by

$$M(s_1,s_2) = \frac{\alpha\Gamma(s_1/2)\Gamma(s_2/2)\Gamma(\beta+s_1/2+s_2/2)}{\pi\Gamma(\beta+1)\Gamma(s_1/2+s_2/2)\alpha^{s_1/2+s_2/2}}$$

Taking the inverse and using the Mellin transform property (3.18) for the H-function yields

$$f_{X,Y}(x,y) = M_2^{-1}[\ M(s_1,s_2) \ ]$$

$$= \frac{\alpha}{\pi\Gamma(\beta+1)} \ {}_1H_{0,1,0,1,1,1}^{1,0,1,0,1,0} \left[\begin{array}{c|c} \sqrt{\alpha} \ x \\ \sqrt{\alpha} \ y \end{array} \ \begin{array}{c} (\beta,1/2) \\ \text{-----} \ ; \ \text{-----} \\ (0,1/2) \\ (0,1/2) \ ; \ (0,1/2) \end{array} \right] \qquad (4.6)$$

(v) Kellogg-Barnes Type II distribution

$$f_{X,Y}(x,y) = \beta\alpha^2 \ e^{-\alpha x \ - \ \beta y/x} \ , \qquad \begin{array}{l} x,y>0 \\ \alpha,\beta>0 \end{array}$$

The Mellin transform is given as

$$M(s_1,s_2) = \beta\alpha^2 \int_0^\infty y^{s_2-1} \ [\int_0^\infty x^{s_1-1} \ e^{-\alpha x-\beta y/x} \ dx] \ dy$$

Using (95:313 # 17) to evaluate the inner integral yields

$$M(s_1,s_2) = \beta\alpha^2 \int_0^\infty y^{s_2-1} \ 2(\beta y/\alpha)^{s_1/2} \ K_{s_1}[\ 2(\alpha\beta y)^{1/2} \ ] \ dy$$

$$= 2\beta\alpha^2(\beta/\alpha)^{s_1/2} \int_0^\infty y^{s_2+s_1/2-1} \ K_{s_1}[\ 2(\alpha\beta y)^{1/2} \ ] \ dy$$

$$= 4\beta\alpha^2(\beta/\alpha)^{s_1/2} \int_0^\infty u^{s_1+2s_2-1} \ K_{s_1}[\ 2\sqrt{\alpha\beta} \ u \ ] \ du$$

where K_ν is the modified Bessel function as defined by Erdélyi

(95:371). Using (95:331 # 26) to complete the integration yields

$$M(s_1,s_2) = \beta\alpha^2 \; \alpha^{-s_1-s_2} \; \beta^{-s_2} \; \Gamma(s_1+s_2)\Gamma(s_2)$$

Inverting and using the H-function definition (3.6) the H-function

form for $f_{X,Y}(x,y)$ is obtained.

$$f_{X,Y}(x,y) = M_2^{-1}[\; M(s_1,s_2)\;]$$

$$= \beta\alpha^2 \; {}_1H^{0,0,1,0,1,0}_{0,0,0,1,0,1} \left[\begin{array}{c|c} & (0,1) \\ \alpha x & \text{-----} \; ; \; \text{-----} \\ \alpha\beta y & \text{-----} \\ & \text{-----} \; ; \; (0,1) \end{array} \right] \qquad (4.7)$$

(vi) <u>Kellogg-Barnes Type III distribution</u>

$$f_{X,Y}(x,y) = \frac{\beta\alpha^c}{\Gamma(c)} \; x^c \; e^{-\alpha x - \beta xy} \;, \qquad \begin{array}{c} x,y>0 \\ \alpha,\beta>0 \;,\; c>2 \end{array}$$

The Mellin transform is given by

$$M(s_1,s_2) = \frac{\beta\alpha^c}{\Gamma(c)} \int_0^\infty x^{s_1+c-1} e^{-\alpha x} [\int_0^\infty y^{s_2-1} e^{-\beta xy} \; dy]\; dx$$

$$= \frac{\alpha^c\beta^{1-s_2}}{\Gamma(c)} \; \Gamma(s_2) \int_0^\infty x^{s_1-s_2+c-1} e^{-\alpha x} \; dx$$

$$= \frac{\beta}{\Gamma(c)} \; \Gamma(s_2)\Gamma(c+s_1-s_2)(\beta/\alpha)^{-s_2} \; \alpha^{-s_1}$$

Inverting and using definition (3.12) or using the Mellin transform

property (3.20) for the H-function, the H-function form for $f_{X,Y}(x,y)$

is obtained.

$$f_{X,Y}(x,y) = M_2^{-1}[\ M(s_1,s_2)\ ]$$

$$= \frac{\beta}{\Gamma(c)}\ 2^{H_{0,0,1,0,0,1}^{0,0,0,1,1,0}}\left[\begin{array}{c|c} & (c,1) \\ \alpha x & \text{-----} \ ; \ (1,1) \\ \frac{\beta}{\alpha}\ y & \\ & \text{-----} \\ & \text{-----} \ ; \ \text{-----} \end{array}\right] \qquad (4.8)$$

4.4 Moments of the H-function Distribution:

Carter (3) showed that the moments of the univariate

H-function distribution can be found by taking the appropriate

derivatives of the characteristic function of the univariate

H-function distribution. To use this approach for the bivariate

H-function distribution, the double Fourier transform of the bivariate

H-function must first be shown to exist. To do this an extension to

Prasad's theorems (57) must be developed. Fortunately, while the

above is considered beyond the scope of this dissertation, there is a

simpler method for obtaining the moment: of the bivariate H-function

distribution.

Using the notation of section 1.3, the n1,n2 ordered

noncentral moment for $f_{X,Y}(x,y)$, $\alpha_{n1,n2}$, is defined by

$$\alpha_{n1,n2} = E[\ x^{n1}y^{n2}\ ]$$

$$= \int_0^\infty \int_0^\infty x^{n1}\ y^{n2}\ f_{X,Y}(x,y)dxdy$$

where E is the expected value operator. From the definition of the Mellin transform, it is clear that

$$M_{s_1,s_2}\{\ f_{X,Y}(x,y)\ \} = E[\ x^{s_1-1}\ y^{s_2-1}\ ]$$

for distributions defined for $x,y > 0$. The n1,n2 ordered moment for $f_{X,Y}(x,y)$ may then be obtained from the Mellin transform of the probability density function. Specifically,

$$\alpha_{n1,n2} = M_{s_1,s_2}\{\ f_{X,Y}(x,y)\ \}\ \Big|_{\substack{s_1=n1+1 \\ s_2=n2+1}}$$

Then from the Mellin transform property (3.18) of the bivariate H-function, the n1,n2 ordered moment for $k_1H[g_1x,g_2y]$ is given by

$$\alpha_{n1,n2} = \frac{k}{g_1^{n1+1}g_2^{n2+1}}\ x_1(n1+1)x_2(n2+1)x_3(n1+n2+2) \qquad (4.9)$$

where $x_1(u)$, $x_2(v)$, and $x_3(u+v)$ are defined by (3.7), (3.8), and (3.9) respectively.

Similarly, from the Mellin transform property (3.20) of the bivariate H-function, the n1,n2 ordered moment for $k_2H[g_1x,g_2y]$ is given by

$$\alpha_{n1,n2} = \frac{k}{g_1^{n1+1} g_2^{n2+1}} x_1(n1+1)x_2(-n2-1)x_3(n1-n2) \qquad (4.10)$$

where $x_1(u)$, $x_2(-v)$, and $x_3(u-v)$ are defined by (3.7), (3.4), and (3.19) respectively.

Following the procedures outlined in section 1.3, (4.9) and (4.10) above can be used to find μ_x, μ_y, μ_{xy}, σ_x^2, and σ_y^2. From these values, the covariance and correlation for X and Y may be found.

Example 4.1: From (4.3), McKay's bivariate gamma distribution can be represented as a bivariate H-function distribution.

$$f_{X,Y}(x,y) = \frac{a^2}{\Gamma(p)} \; {}_1H \begin{matrix} 1,0,0,0,1,0 \\ 1,1,0,0,0,1 \end{matrix} \left[\begin{matrix} ax \\ ay \end{matrix} \middle| \begin{matrix} (p+q-2,1) \\ (p+q-1,1) \; ; \; \text{-----} \\ \text{-----} \\ (p-1,1) \; ; \; \text{-----} \end{matrix} \right]$$

Using (4.9), μ_x may be found by setting n1 = 1 and n2 = 0.

$$\mu_x = \alpha_{1,0} = \frac{a^2}{a^3} \frac{\Gamma(p+1)\Gamma(p+q+1)}{\Gamma(p)\Gamma(p+q+1)}$$

$$= p/a$$

Similarly,

$$\mu_y = \alpha_{0,1} = \frac{a^2}{a^3} \frac{\Gamma(p)\Gamma(p+q+1)}{\Gamma(p)\Gamma(p+q)}$$

$$= (p+q)/a$$

Using the identities $\sigma_x^2 = \alpha_{2,0} - \alpha_{1,0}^2$ and $\sigma_y^2 = \alpha_{0,2} - \alpha_{0,1}^2$, the variances may be found.

$$\sigma_x^2 = \frac{a^2}{a^4} \frac{\Gamma(p+2)\Gamma(p+q+2)}{\Gamma(p)\Gamma(p+q+2)} - (p/a)^2$$

$$= \frac{p(p+1)}{a^2} - \frac{p^2}{a^2}$$

$$= p/a^2$$

Similarly,

$$\sigma_y^2 = \frac{a^2}{a^4} \frac{\Gamma(p)\Gamma(p+q+2)}{\Gamma(p)\Gamma(p+q)} - \frac{(p+q)^2}{a^2}$$

$$= \frac{(p+q+1)(p+q)}{a^2} - \frac{(p+q)^2}{a^2}$$

$$= (p+q)/a^2$$

Using (1.10), the covariance may be found by

$$cov(x,y) = \alpha_{1,1} - \alpha_{1,0}\,\alpha_{0,1}$$

$$= \frac{a^2}{a^4} \frac{\Gamma(p+1)\Gamma(p+q+2)}{\Gamma(p)\Gamma(p+q+1)} - \frac{p(p+q)}{a^2}$$

$$= p/a^2$$

Finally, the correlation is equal to the covariance divided by the standard deviations for x and y.

1.0

1.1

1.25

.45
.50
.56

2.8
3.2
3.6
4.0

2.5
2.2
2.0
1.8

1.4

1.6
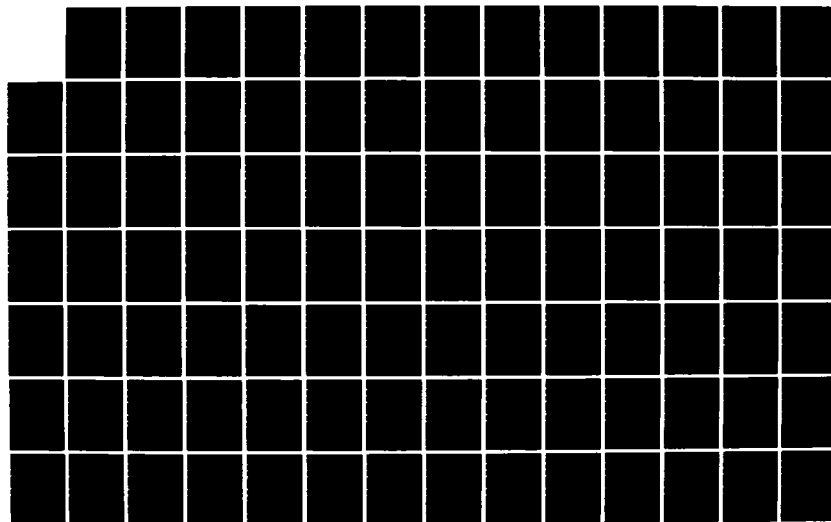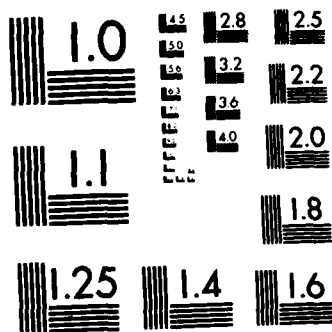
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$$\rho(x,y) = cov(x,y)/\sigma_x\sigma_y$$

$$= \frac{p}{a^2} \frac{a}{\sqrt{p}} \frac{a}{\sqrt{p+q}}$$

$$= \sqrt{p}/\sqrt{p+q}$$

The moments computed above agree with those given by Mardia (47:89) for the bivariate gamma distribution. The moments for all six of the special cases of the H-function distribution given in section 4.3 were computed using (4.9) or (4.10) and agreed with those given by Mardia or computed using the equations given in section 4.4.

4.5 Cumulative Distribution Functions:

4.5.1 Univariate Cumulative Distribution

The cumulative distribution function $H_c(z)$ of a probability density function $H(z)$ is defined as

$$H_c(z) = \int_0^z H(u)du$$

Using a well-known Mellin transform relation from Erdelyi h95:307), Eldred (7:139) derived an expression for $H_c(z)$. Cook (5:103) improved on this expression and derived the form

$$H_c(z) = 1 - (k/c) H_{P+1,Q+1}^{M+1,N}\left[cz \left| \begin{array}{l} (\theta_j+\Theta_j,\Theta_j),(1,1); \\ (0,1),(\phi_j+\Phi_j,\Phi_j) \end{array} \right. \right] \qquad (4.11)$$

Cook then showed a second formulation for the cumulative distribution and proved that the cumulative distribution function for an H-function distribution is another H-function. Using the Laplace transform, Cook derived the following:

$$
H_c(z) = \begin{cases}
(k/c) \; H_{P+1,Q+1}^{M,N+1} \left[cz \; \middle| \; \begin{array}{l} (1,1),(\theta_j+\Theta_j,\Theta_j); \\ (\phi_j+\Phi_j,\Phi_j),(0,1) \end{array} \right] \\
\qquad\qquad\qquad \text{all } -\phi_j/\Phi_j < 1, \; j=1,\dots,M \\[1em]
(-k/c) \; H_{P+1,Q+1}^{M+1,N} \left[cz \; \middle| \; \begin{array}{l} (\theta_j+\Theta_j,\Theta_j),(1,1); \\ (0,1),(\phi_j+\Phi_j,\Phi_j) \end{array} \right] \\
\qquad\qquad\qquad \text{if any } -\phi_j/\Phi_j \geq 1, \; j=1,\dots,M
\end{cases}
$$

$$(4.12)$$

$H_c(z)$ can also be computed using the Mellin transform property for integrals given by Oberhettinger (15:12) and Sneddon (106:269).

$$
M_s \left\{ \int_0^z f(u)\,du \right\} = \frac{1}{s} M_{s+1} \{ f(z) \}
$$

Letting $-1/s = \Gamma(-s)/\Gamma(1-s)$ results in (4.12) case I and letting $-1/s = -\Gamma(s)/\Gamma(s+1)$ results in (4.12) case II.

Comparing (4.11) to (4.12) case II, it appears at first glance that one of the two formulations must be in error. The Laplace transform introduced by Carter (3) can result in one or more of the poles associated with the (ϕ_j,Φ_j) overlapping with the poles

associated with the new $(0,1)$ term in the numerator. Under such
conditions, no contour exists to properly separate the poles. To
correct this problem, Cook $(5:81)$ eliminates the overlap by using an
equivalent expression for the $\Gamma(1-s)$ term introduced in the
development of the Laplace transform of the H-function. The
replacement of this equivalent identity results in (4.12) case II.

Assume $-\phi_j/\Phi_j \geq 1$ for some j. Then (4.12) case II is valid
and (4.11) is not valid. Further, (4.11) is only invalid if the poles
associated with $\Gamma(1-\theta_j-\Theta_j-\Theta_j s)$ overlap with the poles associated with
the new term $\Gamma(s)$. However, these poles will only overlap if
$(\theta_j+\Theta_j) \geq 1$ for some j, $j = 1,\ldots,N$. Now consider the poles of the
density function. If this condition holds, then some of the poles of
$\Gamma(\phi_j+\Phi_j s)$, (for $-\phi_j/\Phi_j$), will occur at values of $s \geq 1$ and some of the
poles of $\Gamma(1-\theta_j-\Theta_j s)$, (for $\theta_j+\Theta_j \geq 1$), will occur at values of $s \leq 1$.
By definition (3.1) such an overlap of poles is not allowable.
Therefore, the condition of $-\phi_j/\Phi_j$ will never occur in H-function
distributions and the cumulative distribution is given by (4.12) case
I only. These results are summarized in the following theorem.

THEOREM 4.1: If $kH(cz)$ is an H-function probability density function
as defined by (4.1), then $-\phi_j/\Phi_j < 1$, $j = 1,\ldots,M$.

From the discussion above, it would seem that it should also
be true, for the same reasons, that $(\theta_j+\Theta_j) < 1$. While a check of all
the special cases listed by Cook, $(6:85-87)$, support this idea, it is

not possible to go from (4.11) to (4.12) and prove that this is true in general.

It should also be pointed out that the improved Laplace transform for the H-function as given by Cook (6:82) is not in error. Specifically, it does not necessarily hold that $-\phi_j/\phi_j < 1$, $j = 1,\ldots,N$, for the H-function in general. It is only true if the H-function is a density function.

4.5.2 Bivariate Cumulative Distribution

The cumulative distribution function $H_c[x,y]$ associated with the probability density function, $H[x,y]$ is defined by

$$H_c[x,y] = \int_0^x \int_0^y H[u,v] \, du \, dv$$

$H_c[x,y]$ can be obtained by direct integration or through the use of the Mellin transform of $H[x,y]$. The latter method is most often preferable as it avoids the necessity of evaluating $H[x,y]$ in order to derive $H_c[x,y]$.

The use of the preceding procedures in evaluating $H_c[x,y]$ is made possible by noting from equation (2.14) that

$$M_{s_1,s_2}\{ 1-H_c[x,y] \} = (s_1 s_2)^{-1} M_{s_1+1, s_2+1}\{ H[x,y] \}$$

Inverting the equation above and subtracting one from both sides yields

$$H_c[x,y] = 1 - M_2^{-1}[(s_1 s_2)^{-1} M_{s_1+1,s_2+1}\{ H[x,y] \}]$$

Consider first $_1H[x,y]$. Substituting the Mellin transform for $_1H[x,y]$ given by (3.18) into the equation above and replacing s_1 by s_1+1 and s_2 by s_2+1 yields

$$_1H_c[x,y] = 1 - M_2^{-1}[\frac{k}{g_1 g_2 s_1 s_2} x_1(s_1+1)x_2(s_2+2)x_3(s_1+s_2+2)g_1^{-s_1}g_2^{-s_2}]$$

where $x_1(u)$, $x_2(v)$, and $x_3(u+v)$ are defined by (3.7), (3.8), and (3.9) respectively. Recognizing that

$$\frac{1}{s_1 s_2} = \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+1)\Gamma(s_2+1)}$$

Substituting this identity into the inversion above yields

$$_1H_c[x,y] = 1 -$$
$$\frac{k}{g_1 g_2} M_2^{-1}\left[\frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+1)\Gamma(s_2+1)} x_1(s_1+1)x_2(s_2+1)x_3(s_1+s_2+2)g_1^{-s_1}g_2^{-s_2} \right]$$

Completing the inversion and using the bivariate H-function definition (3.6), the cumulative H-function form may be found.

$$_1H_c[x,y] = 1 -$$

$$K_1 H^{M_1+1,N_1,M_2+1,N_2,M_3,N_3}_{P_1+1,Q_1+1,P_2+1,Q_2+1,P_3,Q_3} \left[\begin{matrix} g_1x \\ g_2y \end{matrix} \middle| \begin{matrix} (e_1+2E_1,E_1) \\ (a_1+A_1,A_1),(1,1);(c_1+C_1,C_1),(1,1) \\ (f_1+2F_1,F_1) \\ (0,1),(b_1+B_1,B_1);(0,1),(d_1+D_1,D_1) \end{matrix} \right]$$

where $K = k/g_1g_2$ (4.13)

Using property (3.16), $m = n = 1$, equation (4.13) can be written as

$$_1H_c[x,y] = 1 -$$

$$K_1 H^{M_1+1,N_1,M_2+1,N_2,M_3,N_3}_{P_1+1,Q_1+1,P_2+1,Q_2+1,P_3,Q_3} \left[\begin{matrix} g_1x \\ g_2y \end{matrix} \middle| \begin{matrix} (e_1,E_1) \\ (a_1,A_1),(0,1) ; (c_1,C_1),(0,1) \\ (f_1,F_1) \\ (-1,1),(b_1,B_1) ; (-1,1),(d_1,D_1) \end{matrix} \right]$$

where $K = kxy$ (4.14)

Using the same procedure outlined above, a similar
representation can be obtained for the cumulative distribution
function for $_2H[x,y]$.

$$_2H_c[x,y] = 1 -$$

$$K_2H^{M_1+1,N_1,M_2,N_2+1,M_3,N_3}_{P_1+1,Q_1+1,P_2+1,Q_2+1,P_3,Q_3} \left[\begin{array}{c|c} g_1x & (e_i,E_i) \\ & (a_i+A_i,A_i),(1,1);(1,1),(c_i-C_i,C_i) \\ g_2y & (f_i,F_i) \\ & (0,1),(b_i+B_i,B_i);(d_i-D_i,D_i),(0,1) \end{array} \right]$$

where $K = k/g_1g_2$ (4.15)

Using property (3.17), $m = n = 1$, equation (4.15) can be written as

$$_2H_c[x,y] = 1 -$$

$$K_2H^{M_1+1,N_1,M_2,N_2+1,M_3,N_3}_{P_1+1,Q_1+1,P_2+1,Q_2+1,P_3,Q_3} \left[\begin{array}{c|c} g_1x & (e_i,E_i) \\ & (a_i,A_i),(0,1) \; ; \; (2,1),(c_i,C_i) \\ g_2y & (f_i,F_i) \\ & (-1,1),(b_i,B_i) \; ; \; (d_i,D_i),(1,1) \end{array} \right]$$

where $K = kxy$ (4.16)

As indicated by (4.14) and (4.16), $_rH_c[x,y]$ can be found at
the same time as $_rH[x,y]$ by using the calculations for the residues of
$_rH[x,y]$. For $_1H[x,y]$, multiply each residue in the s_2 plane by
$1/(s_2-1)$, and then add the pole $s_2=1$ (or increasing by 1 the order of
an existing pole at $s_2=1$). For $_2H[x,y]$, multiply each residue in the
s_2 plane by $1/(-1-s_2)$, and then add the pole $s_2=-1$ (or increasing by 1
the order of an existing pole at $s_2=-1$). For both $_1H[x,y]$ and
$_2H[x,y]$, multiply each residue in the s_1 plane by $1/(s_1-1)$, and then
add the pole $s_1=1$.

Another formula for the cumulative distribution function of an H-function probability density can be derived using the Mellin transform.

THEOREM 4.2: The bivariate cumulative distribution function for a bivariate H-function probability density function is a bivariate H-function.

Extending the Mellin transform for integrals, (15:12;106:269), to two variables and letting $-1/s_1 = \Gamma(-s_1)/\Gamma(1-s_1)$ and $-1/s_2 = \Gamma(-s_2)/\Gamma(1-s_2)$ yields the bivariate Mellin transform property

$$M_{s_1 s_2}\{\int_0^x \int_0^y f(u,v)dudv\} = \frac{\Gamma(-s_1)\Gamma(-s_2)}{\Gamma(1-s_1)\Gamma(1-s_2)} M_{s_1+1,s_2+1}\{f(x,y)\} \qquad (4.17)$$

This property may also be obtained as a simple perturbation of property (2.14).

Applying (4.17) and (3.18) to $_1H[x,y]$ yields

$$_1H_c[x,y] =$$

$$M_2^{-1}\left[\frac{k\Gamma(-s_1)\Gamma(-s_2)}{g_1g_2\Gamma(1-s_1)\Gamma(1-s_2)} \quad x_1(s_1+1)x_2(s_2+1)x_3(s_1+s_2+2)g_1^{-s_1}g_2^{-s_2}\right]$$

$$= K_1 H^{M_1,N_1+1,M_2,N_2+1,M_3,N_3}_{P_1+1,Q_1+1,P_2+1,Q_2+1,P_3,Q_3} \left[\begin{matrix} g_1 x \\ g_2 y \end{matrix} \middle| \begin{matrix} (e_1+2E_1,E_1) \\ (1,1),(a_1+A_1,A_1);(1,1),(c_1+C_1,C_1) \\ (f_1+2F_1,F_1) \\ (b_1+B_1,B_1),(0,1);(d_1+D_1,D_1),(0,1) \end{matrix} \right]$$

where $K = k/g_1 g_2$ (4.18)

$_1 H_c[x,y]$ may also be obtained by reversing the order of integration of the cumulative distribution function integral and the bivariate H-function contour integral. Let

$$H[x,y] = k \frac{1}{(2\pi i)^2} \iint (\sim)(g_1 x)^{-s_1} (g_2 y)^{-s_2} ds_1 ds_2$$

where $\iint$ represents the bivariate H-function contour integrals as defined by (3.6) and $(\sim)$ represents the gamma products in the bivariate H-function definition (3.6) which do not depend on the variables x and y. Assuming convergence of the H-function, then

$$_1 H_c[x,y] = \int_0^x \int_0^y k \frac{1}{(2\pi i)^2} \iint (\sim)(g_1 u)^{-s_1}(g_2 v)^{-s_2} ds_1 ds_2 du dv$$

$$= k \frac{1}{(2\pi i)^2} \iint (\sim)[\int_0^x (g_1 u)^{-s_1} \int_0^y (g_2 v)^{-s_2} dv du] ds_1 ds_2$$

$$= k \frac{1}{(2\pi i)^2} \iint (\sim) xy \frac{(g_1 x)^{-s_1}(g_2 y)^{-s_2}}{(1-s_1)(1-s_2)} ds_1 ds_2 \qquad (4.19)$$

replacing $[(1-s_1)(1-s_2)]^{-1}$ by $\Gamma(1-s_1)\Gamma(1-s_2)/\Gamma(2-s_1)\Gamma(2-s_2)$ and using (3.6), (4.19) may be expressed as

$$_1H_c[x,y] =$$

$$K_1H^{M_1,N_1+1,M_2,N_2+1,M_3,N_3}_{P_1+1,Q_1+1,P_2+1,Q_2+1,P_3,Q_3}\left[\begin{array}{c|c} & (e_i,E_i) \\ g_1x & (0,1),(a_i,A_i) \; ; \; (0,1),(c_i,C_i) \\ g_2y & (f_i,F_i) \\ & (b_i,B_i),(-1,1) \; ; \; (d_i,D_i),(-1,1) \end{array}\right]$$

where $K = kxy$ $\hspace{4cm}$ (4.20)

Using property (3.16), $m = n = 1$, (4.20) may be expressed as (4.18).

Using similar procedures, the cumulative distribution function $_2H_c[x,y]$ for $_2H[x,y]$ may be found and is also given as a bivariate H-function. Using property (4.17) and the Mellin transform for $_2H[x,y]$, (3.20), $_2H_c[x,y]$ is given as

$$_2H_c[x,y] =$$

$$M_2^{-1}\left[\frac{k\Gamma(-s_1)\Gamma(-s_2)}{g_1g_2\Gamma(1-s_1)\Gamma(1-s_2)} \; x_1(s_1+1)x_2(-s_2-1)x_3(s_1-s_2)g_1^{-s_1}g_2^{-s_2}\right]$$

Replacing s_2 by $-s_2$ and completing the inversion by use of (3.12), the cumulative distribution function for $_2H[x,y]$ may be found and is given by

$$_2H_c[x,y] =$$

$$= K_2H^{M_1,N_1+1,M_2+1,N_2,M_3,N_3}_{P_1+1,Q_1+1,P_2+1,Q_2+1,P_3,Q_3}\left[\begin{matrix} g_1x \\ g_2y \end{matrix} \middle| \begin{matrix} (e_i,E_i) \\ (1,1),(a_i+A_i,A_i);(c_i-C_i,C_i),(1,1) \\ (f_i,F_i) \\ (b_i+B_i,B_i),(0,1);(0,1),(d_i-D_i,D_i) \end{matrix}\right]$$

where $K = k/g_1g_2$ \hfill (4.21)

Using property (3.17), (4.21) may be expressed as

$$_2H_c[x,y] =$$

$$= K_2H^{M_1,N_1+1,M_2+1,N_2,M_3,N_3}_{P_1+1,Q_1+1,P_2+1,Q_2+1,P_3,Q_3}\left[\begin{matrix} g_1x \\ g_2y \end{matrix} \middle| \begin{matrix} (e_i,E_i) \\ (0,1),(a_i,A_i)\ ;\ (c_i,C_i),(2,1) \\ (f_i,F_i) \\ (b_i,B_i),(-1,1)\ ;\ (1,1),(d_i,D_i) \end{matrix}\right]$$

where $K = kxy$ \hfill (4.22)

Bivariate Gamma Cumulative Distribution: Applying (4.18) to the bivariate gamma probability density function given by (4.3),

$$_1H_c[x,y] = \frac{a^{p+q}}{\Gamma(p)\Gamma(q)} \int_0^x \int_0^y u^{p-1}(v-u)^{q-1}e^{-av}\,dudv$$

$$= \frac{1}{\Gamma(p)} \, {}_1H^{1,1,0,1,1,0}_{2,2,1,1,0,1} \left[\begin{array}{c|c} ax & (p+q,1) \\ & (1,1),(p+q,1) \; ; \; (1,1) \\ ay & \text{-----} \\ & (p,1),(0,1) \; ; \; (0,1) \end{array} \right]$$

$$y>x>0 \; , \; a,p,q>0 \qquad\qquad (4.23)$$

<u>Bivariate</u> <u>Beta</u> <u>Cumulative</u> <u>Distribution</u>: Applying (4.18) to the bivariate beta probability density function given by (4.4),

$$_1H_c[x,y] = \frac{\Gamma(p_1+p_2+p_3)}{\Gamma(p_1)\Gamma(p_2)\Gamma(p_3)} \int_0^x \int_0^y u^{p_1-1} \, v^{p_2-1} \, (1-u-v)^{p_3-1} \, dudv$$

$$= \frac{\Gamma(p_1+p_2+p_3)}{\Gamma(p_1)\Gamma(p_2)} \, {}_1H^{1,1,1,1,0,0}_{1,2,1,2,1,0} \left[\begin{array}{c|c} x & \text{-- ---} \\ & (1,1) \; ; \; (1,1) \\ y & (p_1+p_2+p_3,1) \\ & (p_1,1),(0,1) \; ; \; (p_2,1),(0,1) \end{array} \right]$$

$$x,y>0, \; x+y\leq1, \; p_1,p_2,p_3>0 \qquad (4.24)$$

<u>Quarter</u> <u>Cauchy</u> <u>Cumulative</u> <u>Distribution</u>: Applying (4.18) to the quarter cauchy probability density function given by (4.5),

$$_1H_c[x,y] = \frac{2c}{\pi} \int_0^x \int_0^y (c^2 + u^2 + v^2)^{-3/2} \, dudv$$

$$= \frac{1}{\pi^{3/2}} \; {}_1H^{1,1,1,1,0,1}_{1,2,1,2,1,0} \left[\begin{array}{c} \frac{1}{c}\,x \\[4pt] \frac{1}{c}\,y \end{array} \;\middle|\; \begin{array}{c} ----- \\ (1,1)\ ;\ (1,1) \\ (1/2,1/2) \\ (1/2,1/2),(0,1)\ ;\ (1/2,1/2),(0,1) \end{array} \right]$$

$$x,y>0, \;\; c>0 \tag{4.25}$$

Kellogg-Barnes I Cumulative Distribution: Applying (4.18) to the Kellogg-Barnes I probability density function given by (4.6),

$$_1H_c[x,y] = \frac{4\alpha^{\beta+1}}{\pi\Gamma(\beta+1)} \int_0^x \int_0^y (u^2 + v^2)^\beta \; e^{-\alpha(u^2 + v^2)} \; du\,dv$$

$$= \frac{1}{\pi\Gamma(\beta+1)} \; {}_1H^{1,1,1,1,1,0}_{1,2,1,2,1,1} \left[\begin{array}{c} \sqrt{\alpha}\,x \\[4pt] \sqrt{\alpha}\,y \end{array} \;\middle|\; \begin{array}{c} (\beta+1,1/2) \\ (1,1)\ ;\ (1,1) \\ (1,1/2) \\ (1/2,1/2),(0,1)\ ;\ (1/2,1/2),(0,1) \end{array} \right]$$

$$x,y>0, \;\; \alpha,\beta>0 \tag{4.26}$$

Kellogg-Barnes II Cumulative Distribution: Applying (4.18) to the Kellogg-Barnes II probability density function given by (4.7),

$$_1H_c[x,y] = \beta\alpha^2 \int_0^x \int_0^y e^{-(\alpha u + \beta v/u)} \, du\, dv$$

$$= {_1H}_{1,1,1,2,0,1}^{0,1,1,1,1,0} \left[\begin{array}{c|c} & (2,1) \\ \alpha x & (1,1) \; ; \; (1,1) \\ \alpha\beta y & ----- \\ & (0,1) \; ; \; (1,1),(0,1) \end{array} \right]$$

$$x,y>0, \quad \alpha,\beta>0 \qquad\qquad (4.27)$$

Kellogg-Barnes III Cumulative Distribution: Applying (4.21) to the

Kellogg-Barnes III probability density function given by (4.8),

$$_1H_c[x,y] = \frac{\beta\alpha^c}{\Gamma(c)} \int_0^x \int_0^y u^c \, e^{-(\alpha u + \beta uv)} \, du\, dv$$

$$= \frac{1}{\Gamma(c)} \; {_2H}_{1,1,2,1,0,1}^{0,1,1,1,1,0} \left[\begin{array}{c|c} & (c,1) \\ \alpha x & (1,1) \; ; \; (0,1),(1,1) \\ \frac{\beta}{\alpha} y & ----- \\ & (0,1) \; ; \; (0,1) \end{array} \right]$$

$$x,y>0, \quad \alpha,\beta>0, \quad c>2 \qquad\qquad (4.28)$$

4.6 Evaluation of the Bivariate H-function Constant

Carter (3), Eldred (7), and Springer (18) provide special

cases and a definition of the univariate H-function distribution.
However, it was Cook (5) who gave a methodology for determining the
normalization constant k in definition (4.1). A similar procedure can
be applied to the bivariate H-function distribution constant.

One approach to finding k is to investigate $_rH_c[x,y]$ for large
x and y, since

$$\lim_{\substack{x \to \infty \\ y \to \infty}} {_rH_c[x,y]} = \lim_{\substack{x \to \infty \\ y \to \infty}} k(\text{H-function given by (4.18) or (4.21)}) = 1.$$

That is, if $k\,_rH_c[x,y]$ is a proposed H-function probability density,
use (4.18) or (4.21) to find the associated H-function for the
cumulative distribution function, which for large x and y will be 1/k.

While the above method is feasible, it yields a numerical
approximation. An exact method may be found by equating the right
sides of (4.14) and (4.20), for $_1H_c[x,y]$, which immediately yields:

$1/kxy =$

$$
{}_1H^{M_1+1,N_1,M_2+1,N_2,M_3,N_3}_{P_1+1,Q_1+1,P_2+1,Q_2+1,P_3,Q_3} \left[\begin{array}{c|c} g_1x & \begin{array}{c} (e_i,E_i) \\ (a_i,A_i),(0,1) \; ; \; (c_i,C_i),(0,1) \end{array} \\ g_2y & \begin{array}{c} (f_i,F_i) \\ (-1,1),(b_i,B_i) \; ; \; (-1,1),(d_i,D_i) \end{array} \end{array} \right]
$$

$$
+ {}_1H^{M_1,N_1+1,M_2,N_2+1,M_3,N_3}_{P_1+1,Q_1+1,P_2+1,Q_2+1,P_3,Q_3} \left[\begin{array}{c|c} g_1x & \begin{array}{c} (e_i,E_i) \\ (0,1),(a_i,A_i) \; ; \; (0,1),(c_i,C_i) \end{array} \\ g_2y & \begin{array}{c} (f_i,F_i) \\ (b_i,B_i),(-1,1) \; ; \; (d_i,D_i),(-1,1) \end{array} \end{array} \right]
$$

$$
= \frac{1}{(2\pi i)^2} \iint \left[\frac{1}{(s_1-1)(s_2-1)} \; (\sim) + \frac{1}{(1-s_1)(1-s_2)} \; (\sim) \right] ds_1 ds_2 \qquad (4.29)
$$

where $\iint$ represents the appropriate contour integrals as defined in
(3.6) and $(\sim) = x_1(s_1)x_2(s_2)x_3(s_1+s_2)(g_1x)^{-s_1}(g_2y)^{-s_2}$ also defined in
(3.6).

Compare the residues of the two bivariate H-functions in
(4.29). Inverting with respect to s_2, each RHP residue of the first
bivariate H-function has a matching RHP residue of the second
bivariate H-function that is exactly equal but opposite in sign,
except at the residue at $s_2 = 1$. Similarly, each LHP residue of the
second has a matching residue of the first that is exactly equal but
opposite in sign, except at the residue at $s_2 = 1$. Similar results
hold true for inverting with respect to s_1. Therefore, whether (4.29)

is inverted by $LHP(s_1)$, $LHP(s_2)$; $LHP(s_1)$, $RHP(s_2)$; $RHP(s_1)$, $LHP(s_2)$; or $RHP(s_1)$, $RHP(s_2)$, it reduces to only one term on the right side:

$$\frac{1}{kxy} = (+LHP_{s_1} LHP_{s_2} \text{ residue at } s_1 = s_2 = 1)$$

$$= (-LHP_{s_1} RHP_{s_2} \text{ residue at } s_1 = s_2 = 1)$$

$$= (-RHP_{s_1} LHP_{s_2} \text{ residue at } s_1 = s_2 = 1)$$

$$= (+RHP_{s_1} RHP_{s_2} \text{ residue at } s_1 = s_2 = 1)$$

If the bivariate probability density function $_1H[x,y]$ has no pole at $s_1 = 1$ or $s_2 = 1$, then the bivariate cumulative distribution function $_1H_c[x,y]$ has a pole of order 1 at $s_1 = 1$ and $s_2 = 1$ and (4.29) reduces to:

$$\frac{1}{kxy} = \frac{\prod\limits_{1}^{M_1} \Gamma(b_i+B_i) \; \prod\limits_{1}^{N_1} \Gamma(1-a_i-A_i) \; \prod\limits_{1}^{M_2} \Gamma(d_i+D_i)}{\prod\limits_{N_1+1}^{P_1} \Gamma(a_i+A_i) \; \prod\limits_{M_1+1}^{Q_1} \Gamma(1-b_i-B_i) \; \prod\limits_{N_2+1}^{P_2} \Gamma(c_i+C_i)}$$

$$\times \frac{\prod\limits_{1}^{N_2} \Gamma(1-c_i-C_i) \; \prod\limits_{1}^{M_3} \Gamma(e_i+2E_i) \; \prod\limits_{1}^{N_3} \Gamma(1-f_i-2F_i)}{\prod\limits_{M_2+1}^{Q_2} \Gamma(1-d_i-D_i) \; \prod\limits_{N_3+1}^{P_3} \Gamma(f_i+2F_i) \; \prod\limits_{M_3+1}^{Q_3} \Gamma(1-e_i-2E_i)} \frac{1}{g_1 x g_2 y} \quad (4.30)$$

Solving for k and noting the Mellin transform property (3.18),

$$k = \left(1/M_{s_1, s_2}\{ {}_1H[x,y] \} \right) \Bigg|_{\substack{s_1=1 \\ s_2=1}}$$

The above result is summarized in the following theorem.

Theorem 4.3: If ${}_1H[x,y]$ is a bivariate H-function probability density defined by (4.2) for $r = 1$, such that $-b_i/B_i < 1$, $i = 1,\ldots,M_1$, $-d_i/D_i < 1$, $i = 1,\ldots,M_2$, $(a_i+A_i) < 1$, $i = 1,\ldots,N_1$, $(c_i+C_i) < 1$, $i = 1,\ldots,N_2$, (which implies that ${}_1H[x,y]$ has no pole at $s_1 = 1$, $s_2 = 1$), then

$$k = \left(1/M_{s_1, s_2}\{ {}_1H[x,y] \} \right) \Bigg|_{\substack{s_1=1 \\ s_2=1}}$$

$$= g_1 g_2 \frac{\prod_{N_1+1}^{P_1} \Gamma(a_i+A_i) \ \prod_{M_1+1}^{Q_1} \Gamma(1-b_i-B_i) \ \prod_{N_2+1}^{P_2} \Gamma(c_i+C_i)}{\prod_{1}^{M_1} \Gamma(b_i+B_i) \ \prod_{1}^{N_1} \Gamma(1-a_i-A_i) \ \prod_{1}^{M_2} \Gamma(d_i+D_i)}$$

$$\times \frac{\prod_{M_2+1}^{Q_2} \Gamma(1-d_i-D_i) \ \prod_{N_3+1}^{P_3} \Gamma(f_i+2F_i) \ \prod_{M_3+1}^{Q_3} \Gamma(1-e_i-2E_i)}{\prod_{1}^{N_2} \Gamma(1-c_i-C_i) \ \prod_{1}^{M_3} \Gamma(e_i+2E_i) \ \prod_{1}^{N_3} \Gamma(1-f_i-2F_i)} \qquad (4.31)$$

Example 4.2: Consider the probability density function given by

$$f_{X,Y}(x,y) = k e^{-\alpha x - \beta y/x} , \qquad x,y > 0$$

$$= k \; {}_1H^{0,0,1,0,1,0}_{0,0,0,1,0,1} \left[\begin{array}{c|c} \alpha x \\ \alpha \beta y \end{array} \begin{array}{c} (0,1) \\ \text{-----} \; ; \; \text{-----} \\ \text{-----} \\ \text{-----} \; ; \; (0,1) \end{array} \right]$$

from Theorem 4.3, k is given as

$$k = (\alpha)(\alpha\beta) \; \frac{1}{\Gamma(1)\Gamma(2)} = \alpha^2\beta$$

which agrees with the definition of the Kellogg-Barnes II probability

density function.

The five special cases of the bivariate H-function probability

density functions given in section 4.3 all meet the conditions of

Theorem 4.3 and their constants agree with (4.31)

The normalizing constant for $_2H[x,y]$ may be found in the same

fashion by equating the right sides of (4.16) and (4.22),

$1/kxy =$

$$2H^{M_1+1,N_1,M_2,N_2+1,M_3,N_3}_{P_1+1,Q_1+1,P_2+1,Q_2+1,P_3,Q_3} \left[\begin{array}{l} \qquad\qquad\qquad (e_i,E_i) \\ g_1x \quad (a_i,A_i),(0,1) \; ; \; (2,1),(c_i,C_i) \\ g_2y \qquad\qquad\quad (f_i,F_i) \\ \qquad (-1,1),(b_i,B_i) \; ; \; (d_i,D_i),(1,1) \end{array} \right]$$

$$+ \; 2H^{M_1,N_1+1,M_2+1,N_2,M_3,N_3}_{P_1+1,Q_1+1,P_2+1,Q_2+1,P_3,Q_3} \left[\begin{array}{l} \qquad\qquad\qquad (e_i,E_i) \\ g_1x \quad (0,1),(a_i,A_i) \; ; \; (c_i,C_i),(2,1) \\ g_2y \qquad\qquad\quad (f_i,F_i) \\ \qquad (b_i,B_i),(-1,1) \; ; \; (1,1),(d_i,D_i) \end{array} \right]$$

$$= \frac{1}{(2\pi i)^2} \iint \left[\frac{1}{(s_1-1)(-1-s_2)} \; (\sim) + \frac{1}{(1-s_1)(1+s_2)} \; (\sim) \right] ds_1 ds_2 \qquad (4.32)$$

where $\iint$ represents the appropriate contour integrals as defined in (3.6) and $(\sim) = x_1(s_1)x_2(s_2)x_3(s_1+s_2)(g_1x)^{-s_1}(g_2y)^{s_2}$ also defined in (3.6).

Compare the residues of the two H-functions in (4.32). Inverting with respect to s_2, each RHP residue of the second H-function has a matching RHP residue of the first H-function that is exactly equal but opposite in sign, except at the residue $s_2 = -1$. Similarly, each LHP residue of the first has a matching residue of the second that is exactly equal but opposite in sign, except at the residue at $s_2 = -1$. Similar results hold true for inverting with respect to s_1 except that extra pole is at $s_1 = 1$.

If the probability density function $_2H[x,y]$ has no pole at $s_1 = 1$ and $s_2 = -1$, then the cumulative distribution function $_2H_c[x,y]$ has a pole of order 1 at $s_1 = 1$ and $s_2 = -1$ and (4.32) reduces to

$$\frac{1}{kxy} = \frac{\prod\limits_{1}^{M_1} \Gamma(b_i+B_i) \ \prod\limits_{1}^{N_1} \Gamma(1-a_i-A_i) \ \prod\limits_{1}^{M_2} \Gamma(d_i-D_i)}{\prod\limits_{N_1+1}^{P_1} \Gamma(a_i+A_i) \ \prod\limits_{M_1+1}^{Q_1} \Gamma(1-b_i-B_i) \ \prod\limits_{N_2+1}^{P_2} \Gamma(c_i-C_i)}$$

$$\times \ \frac{\prod\limits_{1}^{N_2} \Gamma(1-c_i+C_i) \ \prod\limits_{1}^{M_3} \Gamma(e_i) \ \prod\limits_{1}^{N_3} \Gamma(1-f_i)}{\prod\limits_{M_2+1}^{Q_2} \Gamma(1-d_i+D_i) \ \prod\limits_{N_3+1}^{P_3} \Gamma(f_i) \ \prod\limits_{M_3+1}^{Q_3} \Gamma(1-e_i)} \ \frac{1}{g_1{}^x g_2{}^y} \qquad (4.33)$$

solving for k and noting the Mellin transform property (3.21) yields

$$k = \left(\ 1/M_{s_1,s_2}\{ \ _2H[x,y] \ \} \ \right)\Bigg|_{\substack{s_1=1 \\ s_2=1}}$$

The above result is summarized in the following theorem.

__Theorem__ __4.4:__ If $_2H[x,y]$ is a bivariate H-function probability density defined by (4.2) for $r = 2$, such that $-b_i/B_i < 1$, $i = 1,\ldots,M_1$, $d_i/D_i > 1$, $i = 1,\ldots,M_2$, $(a_i+A_i) < 1$, $i = 1,\ldots,N_1$, $(c_i-C_i) < 1$, $i = 1,\ldots,N_2$, (which implies that $_2H[x,y]$ has no pole at $s_1 = 1$ or $s_2 = -1$), then

$$k = \left.\left(1/M_{s_1,s_2}\{ \,_2H[x,y] \} \right)\right|_{\substack{s_1=1 \\ s_2=1}}$$

$$= g_1 g_2 \frac{\prod_{N_1+1}^{P_1} \Gamma(a_i+A_i) \ \prod_{M_1+1}^{Q_1} \Gamma(1-b_i-B_i) \ \prod_{N_2+1}^{P_2} \Gamma(c_i-C_i)}{\prod_{1}^{M_1} \Gamma(b_i+B_i) \ \prod_{1}^{N_1} \Gamma(1-a_i-A_i) \ \prod_{1}^{M_2} \Gamma(d_i-D_i)}$$

$$x \ \frac{\prod_{M_2+1}^{Q_2} \Gamma(1-d_i+D_i) \ \prod_{N_3+1}^{P_3} \Gamma(f_i) \ \prod_{M_3+1}^{Q_3} \Gamma(1-e_i)}{\prod_{1}^{N_2} \Gamma(1-c_i+C_i) \ \prod_{1}^{M_3} \Gamma(e_i) \ \prod_{1}^{N_3} \Gamma(1-f_i)} \qquad (4.34)$$

<u>Example</u> <u>4.3</u>: Consider the bivariate probability density function given by

$$f_{X,Y}(x,y) = k(\alpha x)^c \ e^{-(\alpha x + \beta xy)} , \qquad x,y>0$$

$$= k \ _2H^{0,0,0,1,1,0}_{0,0,1,0,0,1} \left[\left. \begin{array}{c} \alpha x \\ \frac{\beta}{\alpha} y \end{array} \right| \begin{array}{c} (c,1) \\ \text{-----} \ ; \ (1,1) \\ \text{-----} \\ \text{-----} \ ; \ \text{-----} \end{array} \right]$$

Conditions of Theorem 4.4 are satisfied, and k is given by

$$k = (\frac{\beta}{\alpha})(\alpha) \; \frac{1}{\Gamma(1)\Gamma(c)} = \frac{\beta}{\Gamma(c)}$$

which agrees with constant for the definition of the Kellogg-Barnes II distribution.

Another way to arrive at Theorems (4.3) and (4.4) is to consider the zero-zero moment of a probability density function $k_r H[x,y]$, $x,y > 0$, and $M_{s_1,s_2}\{ _r H[x,y] \}$ has no pole at $s_1 = 1$ or $s_2 = 1$, for $r = 1$, or at $s_1 = 1$ or $s_2 = -1$, for $r = 2$:

$$E[x^0 y^0] = \int\limits_0^\infty \int\limits_0^\infty kH[x,y] \; dxdy = 1$$

$$= M_{s_1,s_2}\{ kH[x,y] \} \Big|_{\substack{s_1=0+1 \\ s_2=0+1}}$$

$$= k \, M_{s_1,s_2}\{ H[x,y] \} \Big|_{\substack{s_1=1 \\ s_2=1}}$$

solving for k yields

$$k = (\; 1/M_{s_1,s_2}\{ H[x,y] \} \;) \Big|_{\substack{s_1=1 \\ s_2=1}}$$

from which Theorems 4.3 and 4.4 immediately follow.

CHAPTER 5

Transformations of H-function Variates

5.1 General Remarks

This chapter has two distinct parts. In the first part, various combinations of products, quotients, and powers of dependent H-function variates are examined. Theorems are presented to show that the product or quotient of two dependent H-function variates is an H-function variate. Powers of dependent H-function variates also result in H-function variates. This result has significance since if the joint density function of two dependent variates can be represented as a bivariate H-function, then the probability density function of a random variable of the form $Z = X^p Y^q$ is given by a univariate H-function and may be inverted using the inversion routine given by Cook (3).

The second part of the chapter examines various combinations of H-function variates from two or more bivariate H-function probability density functions. Theorems are given for products and quotients of variates from bivariate H-function distributions which are pairwise independent. These results are also significant in that by combining these results with the results from part one, a powerful general theory for finding rational combinations of mixtures of dependent and independent random variables now exists.

5.2 Transformations of Dependent H-function Variates

5.2.1 Transformations of $_1H[x,y]$ Variates

As is stated in the following theorem, one of the most significant properties of the bivariate H-function distribution is that the probability distribution of products or quotients of its variates reduces to a univariate H-function. This is similar to results derived by Carter (3) for independent H-function variates, but this theorem shows the result holds for dependent H-function variates as well. It is well known that such a property is not common among the classical distributions.

Theorem 5.1: If X and Y are H-function variates with joint probability density function $f_{X,Y}(x,y)$ where

$$f_{X,Y}(x,y) = \begin{cases} k_1 H[g_1 x, g_2 y] & , x,y > 0 \\ 0 & , \text{otherwise} \end{cases}$$

then the probability density function of the random variable

$$z = X^p Y^q$$

p,q rational, is given by

$$f_Z(z) = \begin{cases} K g_1^{p-1} g_2^{q-1} H_{P\ Q}^{M\ N} \left[g_1^p g_2^q z \ \middle| \ \begin{matrix} \eta_1 \\ \eta_2 \end{matrix} \right] & , z > 0 \\ 0 & \text{otherwise} \end{cases}$$

(5.1)

where for

<u>Case</u> <u>I</u> $p>0$, $q>0$

$$K=k, \quad M=\Sigma M_i, \quad N=\Sigma N_i, \quad P=\Sigma P_i, \quad Q=\Sigma Q_i, \quad i=1,2,3,$$

$$n_1 = (a_i+A_i-A_ip,A_ip), \quad i=1\ldots N_1, \quad (c_i+C_i-C_iq,C_iq), \quad i=1\ldots N_2,$$

$$(f_i+2F_i-F_i(p+q),F_i(p+q)), \quad i=1\ldots N_3,$$

$$(a_i+A_i-A_ip,A_ip), \quad i=N_1+1\ldots P_1, \quad (c_i+C_i-C_iq,C_iq), \quad i=N_2+1\ldots P_2,$$

$$(f_i+2F_i-F_i(p+q),F_i(p+q)), \quad i=N_3+1\ldots P_3$$

$$n_2 = (b_i+B_i-B_ip,B_ip), \quad i=1\ldots M_1, \quad (d_i+D_i-D_iq,D_iq), \quad i=1\ldots M_2,$$

$$(e_i+2E_i-E_i(p+q),E_i(p+q)), \quad i=1\ldots M_3,$$

$$(b_i+B_i-B_ip,B_ip), \quad i=M_1+1\ldots Q_1, \quad (d_i+D_i-D_iq,D_iq), \quad i=M_2+1\ldots Q_2,$$

$$(e_i+2E_i-E_i(p+q),E_i(p+q)), \quad i=M_3+1\ldots Q_3$$

<u>Case</u> <u>II</u> $p>0$, $q<0$, $|p|>|q|$

$$K=k, \quad M=M_1+N_2+M_3, \quad N=N_1+M_2+N_3, \quad P=P_1+Q_2+P_3, \quad Q=Q_1+P_2+Q_3,$$

$$n_1 = (a_i+A_i-A_ip,A_ip), \quad i=1\ldots N_1, \quad (1-d_i-D_i+D_iq,-D_iq), \quad i=1\ldots M_2$$

$$(f_i+2F_i-F_i(p+q),F_i(p+q)), \quad i=1\ldots N_3$$

$$(a_i+A_i-A_ip,A_ip), \quad i=N_1+1\ldots P_1, \quad (1-d_i-D_i+D_iq,-D_iq), \quad i=M_2+1\ldots Q_2$$

$$(f_i+2F_i-F_i(p+q),F_i(p+q)), \quad i=N_3+1\ldots P_3$$

$$n_2 = (b_i+B_i-B_ip,B_ip), \quad i=1\ldots M_1, \quad (1-c_i-C_i+C_iq,-C_iq), \quad i=1,\ldots N_2,$$

$$(e_i+2E_i-E_i(p+q),E_i(p+q)), \quad i=1\ldots M_3,$$

$$(b_i+B_i-B_ip,B_ip), \quad i=M_1+1\ldots Q_1, \quad (1-c_i-C_i+C_iq,-C_iq), \quad i=N_2+1,\ldots P_2,$$

$$(e_i+2E_i-E_i(p+q),E_i(p+q)), \quad i=M_3+1\ldots Q_3,$$

<u>Case</u> <u>III</u> $p>0$, $q<0$, $|p|<|q|$

$$K=k, \quad M=M_1+N_2+N_3, \quad N=N_1+M_2+M_3, \quad P=P_1+Q_2+Q_3, \quad Q=Q_1+P_2+P_3,$$

$$\eta_1 = (a_i+A_i-A_ip,A_ip), \quad i=1\ldots N_1, \quad (1-d_i-D_i+D_iq,-D_iq), \quad i=1\ldots M_2$$

$$(1-e_i-2E_i+E_i(p+q),-E_i(p+q)), \quad i=1\ldots M_3$$

$$(a_i+A_i-A_ip,A_ip), \quad i=N_1+1\ldots P_1, \quad (1-d_i-D_i+D_iq,-D_iq), \quad i=M_2+1\ldots Q_2$$

$$(1-e_i-2E_i+E_i(p+q),-E_i(p+q)), \quad i=M_3+1\ldots Q_3$$

$$\eta_2 = (b_i+B_i-B_ip,B_ip), \quad i=1\ldots M_1, \quad (1-c_i-C_i+C_iq,-C_iq), \quad i=1\ldots N_2,$$

$$(1-f_i-2F_i+F_i(p+q),-F_i(p+q)), \quad i=1\ldots N_3$$

$$(b_i+B_i-B_ip,B_ip), \quad i=M_1+1\ldots Q_1, \quad (1-c_i-C_i+C_iq,-C_iq), \quad i=N_2+1\ldots P_2,$$

$$(1-f_i-2F_i+F_i(p+q),-F_i(p+q)), \quad i=N_3+1\ldots P_3$$

Case IV $p>0$, $q<0$, $|p|=|q|$

$$K = k \frac{\prod\limits_{1}^{M_3} \Gamma(e_i+2E_i) \prod\limits_{1}^{N_3} \Gamma(1-f_i-2F_i)}{\prod\limits_{N_3+1}^{P_3} \Gamma(f_i+2F_i) \prod\limits_{M_3+1}^{Q_3} \Gamma(1-e_i-2E_i)}$$

$$M=M_1+N_2, \quad N=N_1+M_2, \quad P=P_1+Q_2, \quad Q=Q_1+P_2,$$

$$\eta_1 = (a_i+A_i-A_ip,A_ip), \quad i=1\ldots N_1, \quad (1-d_i-D_i+D_iq,-D_iq), \quad i=1\ldots M_2,$$

$$(a_i+A_i-A_ip,A_ip), \quad i=N_1+1\ldots P_1, \quad (1-d_i-D_i+D_iq,-D_iq), \quad i=M_2+1\ldots Q_2,$$

$$\eta_2 = (b_i+B_i-B_ip,B_ip), \quad i=1\ldots M_1, \quad (1-c_i-C_i+C_iq,-C_iq), \quad i=1\ldots N_2,$$

$$(b_i+B_i-B_ip,B_ip), \quad i=M_1+1\ldots Q_1, \quad (1-c_i-C_i+C_iq,-C_iq), \quad i=N_2+1\ldots P_2,$$

Case V $p<0$, $q<0$

$$K=k, \quad M=\Sigma N_i, \quad N=\Sigma M_i, \quad P=\Sigma Q_i, \quad Q=\Sigma P_i, \quad i=1,2,3 ,$$

$$\eta_1 = (1-b_i-B_i+B_ip,-B_ip), \ i=1\ldots M_1, \ (1-d_i-D_i+D_iq,-D_iq), \ i=1\ldots M_2,$$

$$(1-e_i-2E_i+E_i(p+q),-E_i(p+q)), \ i=1\ldots M_3$$

$$(1-b_i-B_i+B_ip,-B_ip), \ i=M_1+1\ldots Q_1,$$

$$(1-d_i-D_i+D_iq,-D_iq), \ i=M_2+1\ldots Q_2,$$

$$(1-e_i-2E_i+E_i(p+q),-E_i(p+q)), \ i=M_3+1\ldots Q_3$$

$$\eta_2 = (1-a_i-A_i+A_ip,-A_ip), \ i=1\ldots N_1, \ (1-c_i-C_i+C_iq,-C_iq), \ i=1\ldots N_2,$$

$$(1-f_i-2F_i+F_i(p+q),-F_i(p+q)), \ i=1\ldots N_3$$

$$(1-a_i-A_i+A_ip,-A_ip), \ i=N_1+1\ldots P_1,$$

$$(1-c_i-C_i+C_iq,-C_iq), \ i=N_2+1\ldots P_2,$$

$$(1-f_i-2F_i+F_i(p+q),-F_i(p+q)), \ i=N_3+1\ldots P_3$$

Proof for Theorem 5.1 From Theorem 2.4

$$f_Z(z) = M_1^{-1}[\ M_{ps-p+1,qs-q+1}\{ \ f_{X,Y}(x,y) \ \} \ ]$$

$$= M_1^{-1}[\ M_{ps-p+1,qs-q+1}\{ \ k_1 H[g_1x,g_2y] \ \} \ ]$$

and from Equation (3.19)

$$f_Z(z) = M_1^{-1}\left[\ k g_1^{-u} g_2^{-v} x_1(u)x_2(v)x_3(u+v) \ \Big|_{\substack{u=ps-p+1 \\ v=qs-q+1}} \ \right]$$

$$f_Z(z) = k g_1^{p-1} g_2^{q-1} \frac{1}{(2\pi i)} \int_{h-i\infty}^{h+i\infty} x_1(u)x_2(v)x_3(u+v)\Big|_{\substack{u=ps-p+1 \\ v=qs-q+1}} (g_1^p g_2^q z)^{-s} \ ds \quad (5.2)$$

from the univariate H-function probability density function

definition, (4.1), and the univariate H-function definition, (3.1),

cases I, II, III, IV, and V immediately follow.

Special Cases

Two special cases of this theorem that are of particular

interest are a simple product and quotient of two dependent variables.

The distribution of $Z = XY$: From Theorem 5.1, $p = q = 1$, (5.1)

reduces to

$$
f_Z(z) = \begin{cases} kH^{M\,N}_{P\,Q}\left[g_1 g_2 z \;\middle|\; \begin{array}{c} \eta_1 \\ \eta_2 \end{array}\right] & ,\ z>0 \\ \\ 0 & \text{otherwise} \end{cases}
\tag{5.3}
$$

where $M=\Sigma M_i$, $N=\Sigma N_i$, $P=\Sigma P_i$, $Q=\Sigma Q_i$, $i=1,2,3$,

$\eta_1 = (a_i,A_i)$, $i=1\ldots N_1$, (c_i,C_i), $i=1\ldots N_2$,

$\quad (f_i,2F_i)$, $i=1\ldots N_3$, (a_i,A_i), $i=N_1+1\ldots P_1$,

$\quad (c_i,C_i)$, $i=N_2+1\ldots P_2$, $(f_i,2F_i)$, $i=N_3+1\ldots P_3$

$\eta_2 = (b_i,B_i)$, $i=1\ldots M_1$, (d_i,D_i), $i=1\ldots M_2$,

$\quad (e_i,2E_i)$, $i=1\ldots M_3$,(b_i,B_i), $i=M_1+1\ldots Q_1$,

$\quad (d_i,D_i)$, $i=M_2+1\ldots Q_2$, $(e_i,2E_i)$, $i=M_3+1\ldots Q_3$

Equation (5.3) above may also be found by applying Equation

(2.23) to the Mellin transform, (3.19), of the bivariate H-function

definition. Recombining like terms and inverting by (3.6), Equation

(5.3) may be obtained directly.

For the special case where $M_3=N_3=P_3=Q_3=0$, the bivariate

H-function probability density function reduces to a product of two

independent univariate H-function probability density functions and

(5.3) reduces to the form given by Carter (3:52) for the distribution

of the product of two independent H-function variates.

The distribution of $Z = X/Y$: From Theorem 5.1, $p = 1$, $q = -1$, (5.1)

reduces to

$$f_Z(z) = \begin{cases} \dfrac{k}{g_2^2} H^{M_1+N_2,N_1+M_2}_{P_1+Q_2,Q_1+P_2} \left[\dfrac{g_1}{g_2} z \ \middle| \ \begin{matrix} \eta_1 \\ \eta_2 \end{matrix} \right] & , \ z>0 \\ \\ 0 & \text{otherwise} \end{cases} \tag{5.4}$$

where

$$K = k \frac{1}{P_3} \frac{\prod\limits_{1}^{M_3} \Gamma(e_i+2E_i) \ \prod\limits_{1}^{N_3} \Gamma(1-f_i-2F_i)}{\prod\limits_{N_3+1}^{P_3} \Gamma(f_i+2F_i) \ \prod\limits_{M_3+1}^{Q_3} \Gamma(1-e_i-2E_i)}$$

$\eta_1 = (a_i,A_i), \ i=1\ldots N_1, \ (1-d_i-2D_i,D_i), \ i=1\ldots M_2,$

$\quad (a_i,A_i), \ i=N_1+1\ldots P_1, \ (1-d_i-2D_i,D_i), \ i=M_2+1\ldots Q_2,$

$\eta_2 = (b_i,B_i), \ i=1\ldots M_1, \ (1-c_i-2C_i,C_i), \ i=1\ldots N_2,$

$\quad (b_i,B_i), \ i=M_1+1\ldots Q_1, \ (1-c_i-2C_i,C_i), \ i=N_2+1\ldots P_2,$

Equation (5.4) may also be found by applying Equation (2.24)

to the Mellin transform, (3.19), of the bivariate H-function

definition. Recombining like terms and inverting by (3.6), Equation

(5.4) may be obtained directly.

For the case $M_3=N_3=P_3=Q_3=0$, (5.4) reduces to the univariate H-function probability density function which is the result of the ratio of two independent H-function variates. For $M_3=N_3=P_3=Q_3=0$, (5.4) agrees with the form given by Carter, (3:62).

Example 5.1: Consider the Kellogg-Barnes II distribution given by

$$f_{X,Y}(x,y) = \beta\alpha^2 \exp(-\alpha x - \beta y/x), \qquad \begin{matrix} x,y>0 \\ \alpha,\beta>0 \end{matrix}$$

$$= \beta\alpha^2 \, {}_1H^{0,0,1,0,1,0}_{0,0,0,1,0,1} \left[\begin{matrix} \alpha x \\ \alpha\beta y \end{matrix} \, \middle| \, \begin{matrix} (0,1) \\ \text{-----} \; ; \; \text{-----} \\ \text{-----} \\ \text{-----} \; ; \; (0,1) \end{matrix} \right]$$

The distribution of the random variable Z=X/Y may be found with Theorem 5.1, p = 1, q = -1, or from (5.4) directly, and is given by

$$f_Z(z) = \frac{\beta\alpha^2}{\alpha^2\beta^2} \, \Gamma(2) H^{0\ 1}_{1\ 0} \left[\frac{1}{\beta} z \, \middle| \, \begin{matrix} (-1,1) \\ \text{------} \end{matrix} \right]$$

$$= \frac{1}{\beta} \, \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} \Gamma(2-s)(\tfrac{1}{\beta} z)^{-s} \, ds \tag{5.5}$$

Using Cook's convergence theorem for the H-function, equation (5.5) may be evaluated using the sum of residues in the right half plane. Poles of the integrand from $\Gamma(2-s)$ occur at $s=2,3,4,\ldots$

Evaluating the residues at these points and summing yields

$$\Sigma \text{ residues} = \frac{\beta^2}{z^2} \sum_{i=0}^{\infty} (-1)^i \frac{(\beta/z)^i}{i!}$$

Recognizing this as the power series for the exponential, the
distribution of $f_Z(z)$ is then given by

$$f_Z(z) = \frac{1}{\beta} \Sigma \text{ residues}$$

$$= \beta z^{-2} e^{-\beta/z}$$

which agrees with the results obtained using the Mellin convolution
integral in Example 1.5.

Example 5.2: Suppose the distribution for the random variable Z=Y/X
is desired for X and Y distributed by the Kellogg-Barnes II
distribution given above. Clearly, Theorem 5.1 won't hold in this
case since it does not allow for cases where p<0 and q>0. However, if
the Mellin transform is used to map x into s_2 and y into s_1 and an
appropriate H-function identity found, then Theorem 5.1, case iv,
holds. The H-function representation for the Kellogg-Barnes II
distribution with such a mapping is given by

$$f_{X,Y}(x,y) = \beta\alpha^2 \; {}_1H \; {}^{1,0,0,0,1,0}_{0,1,0,0,0,1} \left[\begin{array}{c|c} \alpha\beta y & \begin{array}{c} (0,1) \\ \text{-----} \; ; \; \text{-----} \\ \end{array} \\ \alpha x & \begin{array}{c} \text{-----} \\ (0,1) \; ; \; \text{-----} \end{array} \end{array} \right]$$

Applying Theorem 5.1, case iv, the distribution for the random
variable Z=Y/X is

$$f_Z(z) = \frac{\beta\alpha^2}{\alpha^2} \Gamma(2) \; H \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \left[\beta z \; \middle| \; \begin{array}{c} ----- \\ (0,1) \end{array} \right]$$

$$= \beta \; \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} \Gamma(s)(\beta z)^{-s} \, ds \qquad\qquad (5.6)$$

The distribution $f_Z(z)$ can be found by evaluating and summing
the residues of (5.6) in the left half plane. The integrand of (5.6)
has poles at s=0,-1,-2,... Evaluating and summing these residues
yields

$$\Sigma \text{ residues} = \sum_{i=0}^{\infty} \frac{(-1)^i (\beta z)^i}{i!}$$

Realizing that this is the power series for an exponential, the
distribution $f_Z(z)$ may be given as

$$f_Z(z) = \beta \; \Sigma \text{ residues}$$

$$= \beta e^{-\beta z}$$

The distribution $f_Z(z)$ may also be determined directly from the
H-function form of Equation (5.6). Using the special cases given by

Carter (3), the exponential distribution is given directly by (5.6).
The solution is verified by use of the Mellin convolution integral in
Example 1.5.

Example 5.3: Consider the probability density function given by

$$f_{X,Y}(x,y) = \frac{4\alpha^{1+\beta/2}}{\sqrt{\pi}\Gamma(\frac{\beta+1}{2})} \; x^\beta \; e^{-\alpha(x^2 + y^2)} \qquad \begin{array}{l} x,y > 0 \\ \alpha,\beta > 0 \end{array}$$

Viewing the density function above it should be realized that
the function can be expressed in the form $f_X(x)\cdot f_Y(y)$. Restated, X
and Y are independent. While theorems given by Carter (3) may be used
to find the distribution of Z = XY, it is illustrative to treat the
density function above as a joint density and use Theorem 5.1 to find
the density function for the random variable Z = XY.

Applying (2.5), followed by (3.6), to the joint probability
density function above, the H-function representation may be found and
is given by

$$f_{X,Y}(x,y) = \frac{\alpha}{\sqrt{\pi}\Gamma(\frac{\beta+1}{2})} \; {}_1H_{0,1,0,1,0,0}^{1,0,1,0,0,0}\left[\begin{array}{c} \sqrt{\alpha}\,x \\ \sqrt{\alpha}\,y \end{array} \left| \begin{array}{c} ---- \\ ---- \; ; \; ---- \\ ---- \\ (\beta/2,1/2) \; ; \; (0,1/2) \end{array} \right. \right] \qquad (5.7)$$

From Theorem 5.1, case 1, p = q = 1, the distribution of Z = XY is
given by

$$f_Z(z) = \frac{\alpha}{\sqrt{\pi}\Gamma(\frac{\beta+1}{2})} H_{0,2}^{2,0}\left[\alpha z \left|\begin{array}{c} ----- \\ (\beta/2,1/2) \; ; \; (0,1/2) \end{array}\right.\right]$$

$$= \frac{\alpha}{\sqrt{\pi}} \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} \Gamma(\frac{s+1}{2})\Gamma(\frac{s}{2})(\alpha z)^{-s} \, ds \qquad (5.8)$$

for $\beta = 1$. From Legéndre's duplication formula, (8:5 #15)

$$\pi^{-1/2} \Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2}) = 2\Gamma(s)2^{-s}$$

Equation (5.8) can be written as

$$f_Z(z) = 2\alpha \frac{1}{2\pi i} \int_{h-i\infty}^{h+i\infty} \Gamma(s)(2\alpha z)^{-s} \, ds$$

$$= 2\alpha H_{0,1}^{1,0}\left[2\alpha z \left|\begin{array}{c} ----- \\ (0,1) \end{array}\right.\right]$$

which from Carter, (3:45), is just the exponential distribution with parameter $\phi = 1/2\alpha$. Specifically,

$$f_Z(z) = 2\alpha e^{-2\alpha z}$$

The solution may be verified using the convolution integral, Theorem 1.5. For $\beta = 1$,

$$f_Z(z) = \frac{4\alpha^{3/2}}{\sqrt{\pi}} \int_0^{\infty} e^{-\alpha(x^2 + (z/x)^2)} \, dx$$

$$= \frac{4\alpha}{\sqrt{\pi}} \int_0^\infty e^{-(u^2 + (\alpha z/u)^2)} \, du$$

$$= 2\alpha e^{-2\alpha z}$$

which agrees with the solution above using a bivariate H-function probability density function and Theorem 5.1.

5.2.2 Transformations of $_2H[x,y]$ Variates

Theorem 5.1 shows that the probability density function of products or quotients of $_1H[x,y]$ variates is a univariate H-function. The following theorem shows this property is shared by $_2H[x,y]$ variates.

Theorem 5.2: If X and Y are H-function variates with joint probability density function $f_{X,Y}(x,y)$ where

$$f_{X,Y}(x,y) = \begin{cases} k {_2H[g_1 x, g_2 y]} \; , \; x,y > 0 \\ 0 \; , \; \text{otherwise} \end{cases}$$

then the probability density function of the random variable

$$Z = X^p Y^q$$

p,q rational, is given by

$$f_Z(z) = \begin{cases} K g_1^{p-1} \, g_2^{q-1} \, H_{P \; Q}^{M \; N} \left[g_1^p g_2^q z \; \middle| \; \begin{array}{c} n_1 \\ n_2 \end{array} \right] \qquad , \; z > 0 \\ \\ 0 \qquad \qquad \text{otherwise} \end{cases} \qquad (5.9)$$

where for

<u>Case I</u> $p>0$, $q>0$, $p>q$

$K=k$, $M=M_1+N_2+M_3$, $N=N_1+M_2+N_3$, $P=P_1+Q_2+P_3$, $Q=Q_1+P_2+Q_3$

$\eta_1 = (a_i+A_i-A_ip, A_ip)$, $i=1\ldots N_1$, $(1-d_i+D_i-D_iq, D_iq)$, $i=1\ldots M_2$,

$\quad (f_i-F_ip+F_iq, F_ip-F_iq)$, $i=1\ldots N_3$,

$\quad (a_i+A_i-A_ip, A_ip)$, $i=N_1+1\ldots P_1$, $(1-d_i+D_i-D_iq, D_iq)$, $i=M_2+1\ldots Q_2$,

$\quad (f_i-F_ip+F_iq, F_ip-F_iq)$, $i=N_3+1\ldots P_3$

$\eta_2 = (b_i+B_i-B_ip, B_ip)$, $i=1\ldots M_1$, $(1-c_i+C_i-C_iq, C_iq)$, $i=1\ldots N_2$,

$\quad (e_i-E_ip+E_iq, E_ip-E_iq)$, $i=1\ldots M_3$,

$\quad (b_i+B_i-B_ip, B_ip)$, $i=M_1+1\ldots Q_1$, $(1-c_i+C_i-C_iq, C_iq)$, $i=N_2+1\ldots P_2$,

$\quad (e_i-E_ip+E_iq, E_ip-E_iq)$, $i=M_3+1\ldots Q_3$

<u>Case II</u> $p>0$, $q>0$, $p<q$

$K=k$, $M=M_1+N_2+N_3$, $N=N_1+M_2+M_3$, $P=P_1+Q_2+Q_3$, $Q=Q_1+P_2+P_3$

$\eta_1 = (a_i+A_i-A_ip, A_ip)$, $i=1\ldots N_1$, $(1-d_i+D_i-D_iq, D_iq)$, $i=1\ldots M_2$,

$\quad (1-e_i+E_ip-E_iq, -E_ip+E_iq)$, $i=1\ldots M_3$,

$\quad (a_i+A_i-A_ip, A_ip)$, $i=N_1+1\ldots P_1$, $(1-d_i+D_i-D_iq, D_iq)$, $i=M_2+1\ldots Q_2$,

$\quad (1-e_i+E_ip-E_iq, -E_ip+E_iq)$, $i=M_3+1\ldots Q_3$

$\eta_2 = (b_i+B_i-B_ip, B_ip)$, $i=1\ldots M_1$, $(1-c_i+C_i-C_iq, C_iq)$, $i=1\ldots N_2$,

$\quad (1-f_i+F_ip-F_iq, -F_ip+F_iq)$, $i=1\ldots N_3$,

$\quad (b_i+B_i-B_ip, B_ip)$, $i=M_1+1\ldots Q_1$, $(1-c_i+C_i-C_iq, C_iq)$, $i=N_2+1\ldots P_2$,

$\quad (1-f_i+F_ip-F_iq, -F_ip+F_iq)$, $i=N_3+1\ldots P_3$

Case III $p>0$, $q>0$, $p=q$

$$K = k \frac{\prod_1^{M_3} \Gamma(e_i) \; \prod_1^{N_3} \Gamma(1-f_i)}{\prod_{N_3+1}^{P_3} \Gamma(f_i) \; \prod_{M_3+1}^{Q_3} \Gamma(1-e_i)}$$

$M=M_1+N_2$, $N=N_1+M_2$, $P=P_1+Q_2$, $Q=Q_1+P_2$

$\eta_1 = (a_i+A_i-A_ip, A_ip)$, $i=1\ldots N_1$, $(1-d_i+D_i-D_iq, D_iq)$, $i=1\ldots M_2$,

$\quad (a_i+A_i-A_ip, A_ip)$, $i=N_1+1\ldots P_1$, $(1-d_i+D_i-D_iq, D_iq)$, $i=M_2+1\ldots Q_2$

$\eta_2 = (b_i+B_i-B_ip, B_ip)$, $i=1\ldots M_1$, $(1-c_i+C_i-C_iq, C_iq)$, $i=1\ldots N_2$,

$\quad (b_i+B_i-B_ip, B_ip)$, $i=M_1+1\ldots Q_1$, $(1-c_i+C_i-C_iq, C_iq)$, $i=N_2+1\ldots P_2$

Case IV $p>0$, $q<0$

$K=k$, $M=\Sigma M_i$, $N=\Sigma N_i$, $P=\Sigma P_i$, $Q=\Sigma Q_i$, $i=1,2,3$

$\eta_1 = (a_i+A_i-A_ip, A_ip)$, $i=1\ldots N_1$, $(c_i-C_i+C_iq, -C_iq)$, $i=1\ldots N_2$,

$\quad (f_i-F_ip+F_iq, F_ip-F_iq)$, $i=1\ldots N_3$,

$\quad (a_i+A_i-A_ip, A_ip)$, $i=N_1+1\ldots P_1$, $(c_i-C_i+C_iq, -C_iq)$, $i=N_2+1\ldots P_2$,

$\quad (f_i-F_ip+F_iq, F_ip-F_iq)$, $i=N_3+1\ldots P_3$

$\eta_2 = (b_i+B_i-B_ip, B_ip)$, $i=1\ldots M_1$, $(d_i-D_i+D_iq, -D_iq)$, $i=1\ldots M_2$,

$\quad (e_i-E_ip+E_iq, E_ip-E_iq)$, $i=1\ldots M_3$,

$\quad (b_i+B_i-B_ip, B_ip)$, $i=M_1+1\ldots Q_1$, $(d_i-D_i+D_iq, -D_iq)$, $i=M_2+1\ldots Q_2$,

$\quad (e_i-E_ip+E_iq, E_ip-E_iq)$, $i=M_3+1\ldots Q_3$

Proof for Theorem 5.2 From Theorem 2.4

$$f_Z(z) = M_1^{-1}[\ M_{ps-p+1,qs-q+1}\{\ f_{X,Y}(x,y)\ \}\ ]$$

$$= M_1^{-1}[\ M_{ps-p+1,qs-q+1}\{\ k_2H[g_1x,g_2y]\ \}\ ]$$

and from Equation (3.21)

$$f_Z(z) = M_1^{-1}\left[\ kg_1^{-u}g_2^{-v}x_1(u)x_2(-v)x_3(u-v)\ \Bigg|\ \begin{matrix} u=ps-p+1 \\ v=qs-q+1 \end{matrix}\ \right]$$

$$= M_1^{-1}\left[\ kg_1^{-u}g_2^{v}x_1(u)x_2(v)x_3(w)\ \Bigg|\ \begin{matrix} u=ps-p+1 \\ v=-qs+q-1 \\ w=(p-q)s-p+q \end{matrix}\ \right]$$

$$f_Z(z) = kg_1^{p-1}g_2^{q-1}\frac{1}{(2\pi i)}\ \int_{h-i\infty}^{h+i\infty} x_1(u)x_2(v)x_3(w)\ \Bigg|\ (g_1^{p}g_2^{q}z)^{-s}\ ds \quad (5.10)$$
$$\begin{matrix} u=ps-p+1 \\ v=-qs+q-1 \\ w=(p-q)s-p+q \end{matrix}$$

from the univariate H-function probability density function

definition, (4.1), and the univariate H-function definition, (3.1),

cases I, II, III, and IV immediately follow.

Special Cases

A simple product and quotient of $_2H[x,y]$ variates are

presented as special cases of Theorem 5.2.

The distribution of Z = XY: From Theorem 5.2, case III, p = q = 1,

(5.9) reduces to

$$f_Z(z) = \begin{cases} K \; H_{P_1+Q_2,Q_1+P_2}^{M_1+N_2,N_1+M_2} \left[g_1 g_2 z \; \middle| \; \begin{matrix} \eta_1 \\ \eta_2 \end{matrix} \right] & , \; z>0 \\ \\ 0 & \text{otherwise} \end{cases} \tag{5.11}$$

where

$$K = k \; \frac{\prod\limits_{1}^{M_3} \Gamma(e_i) \; \prod\limits_{1}^{N_3} \Gamma(1-f_i)}{\prod\limits_{N_3+1}^{P_3} \Gamma(f_i) \; \prod\limits_{M_3+1}^{Q_3} \Gamma(1-e_i)}$$

$$\eta_1 = (a_i, A_i), \; i=1\ldots N_1, \; (1-d_i, D_i), \; i=1\ldots M_2,$$

$$(a_i, A_i), \; i=N_1+1\ldots P_1, \; (1-d_i, D_i), \; i=M_2+1\ldots Q_2$$

$$\eta_2 = (b_i, B_i), \; i=1\ldots M_1, \; (1-c_i, C_i), \; i=1\ldots N_2,$$

$$(b_i, B_i), \; i=M_1+1\ldots Q_1, \; (1-c_i, C_i), \; i=N_2+1\ldots P_2$$

Equation (5.11) can also be derived directly by applying (3.21) and (2.23) to the bivariate H-function distribution definition, (4.2) and the H-function definition (3.12).

The distribution of $Z = X/Y$: From Theorem 5.2, case IV, $p = 1$, $q = -1$, (5.9) reduces to

$$f_Z(z) = \begin{cases} \dfrac{K}{g_2^2} \; H_{P\;Q}^{M\;N} \left[\dfrac{g_1}{g_2} z \; \middle| \; \begin{matrix} \eta_1 \\ \eta_2 \end{matrix} \right] & , \; z>0 \\ \\ 0 & \text{otherwise} \end{cases} \tag{5.12}$$

where $K=k$, $M=\Sigma M_i$, $N=\Sigma N_i$, $P=\Sigma P_i$, $Q=\Sigma Q_i$, $i=1,2,3$

$\eta_1 = (a_i,A_i)$, $i=1\ldots N_1$, (c_i-2C_i,C_i), $i=1\ldots N_2$,

$\qquad (f_i-2F_i,2F_i)$, $i=1\ldots N_3$, (a_i,A_i), $i=N_1+1\ldots P_1$,

$\qquad (c_i-2C_i,C_i)$, $i=N_2+1\ldots P_2$, $(f_i-2F_i,2F_i)$, $i=N_3+1\ldots P_3$

$\eta_2 = (b_i,B_i)$, $i=1\ldots M_1$, (d_i-2D_i,D_i), $i=1\ldots M_2$,

$\qquad (e_i-2E_i,2E_i)$, $i=1\ldots M_3$, (b_i,B_i), $i=M_1+1\ldots Q_1$,

$\qquad (d_i-2D_i,D_i)$, $i=M_2+1\ldots Q_2$, $(e_i-2E_i,2E_i)$, $i=M_3+1\ldots Q_3$

Equation (5.12) can also be derived directly by applying (3.21) and (2.24) to the bivariate H-function probability density definition (4.2) and to the H-function definition (3.12).

Example 5.4 Consider the Kellogg-Barnes III distribution given by

$$f_{X,Y}(x,y) = \frac{\beta\alpha^c}{\Gamma(c)}x^c \exp(-\alpha x - \beta xy) , \qquad \begin{array}{l} x,y>0 \\ \alpha,\beta>0 , c>2 \end{array}$$

$$= \frac{\beta}{\Gamma(c)} \; 2^{H}{}^{0,0,0,1,1,0}_{0,0,1,0,0,1}\left[\begin{array}{c|c} \alpha x & \begin{array}{c} (c,1) \\ \hline \; ; (1,1) \end{array} \\ \frac{\beta}{\alpha} y & \begin{array}{c} \hline \\ \hline \; ; \; \hline \end{array} \end{array}\right]$$

Applying Theorem 5.2, case III, $p=q=1$, or applying (5.11) directly, the distribution for the random variable $Z=XY$ is

$$f_Z(z) = \frac{\beta}{\Gamma(c)} \ \Gamma(c) \ H \begin{array}{c} 1 \ \ 0 \\ \\ 0 \ \ 1 \end{array} \left[\frac{\alpha\beta}{\alpha} \ z \ \middle| \ \begin{array}{c} ----- \\ (0,1) \end{array} \right]$$

$$= \beta \ H \begin{array}{c} 1 \ \ 0 \\ \\ 0 \ \ 1 \end{array} \left[\beta z \ \middle| \ \begin{array}{c} ----- \\ (0,1) \end{array} \right] \tag{5.13}$$

The distribution of $f_Z(z)$ can be found directly from (5.13) by realizing that (5.13) is the H-function identity for the exponential distribution with parameter $1/\beta$ as given by Carter (3). Then

$$f_Z(z) = \beta e^{-\beta z}$$

which agrees with the solution obtained in Example 1.4 using the convolution integral.

5.3 Transformations for Two or More Bivariate Distributions

In the last section theorems were presented deriving the distribution for a random variable resulting from some combination of dependent H-function variates sharing a bivariate H-function probability density function. This section is concerned with procedures for finding the bivariate probability density functions of random variables which are the products or ratios of other bivariate random variables.

5.3.1 Tranformations for $_1H[x,y]$ Variates

5.3.1.1 The Distribution of Products

Theorem 5.3: If X_1, Y_1; X_2, Y_2; $\ldots$; X_n, Y_n are n pairwise independent random variables with bivariate probability density functions $f_1(x_1, y_1)$, $f_2(x_2, y_2)$, $\ldots$, $f_n(x_n, y_n)$ respectively, where

$$f_j(x_j, y_j) = \begin{cases} k_j \; _1H[g_{1j}x_j, g_{2j}y_j] \; , & x_j, y_j > 0 \\ \\ 0 \; , & \text{otherwise} \end{cases}$$

for $j=1,\ldots,n$, then the bivariate probability density function of the random variables

$$Z = \prod_{j=1}^{n} X_j \quad ; \quad W = \prod_{j=1}^{n} Y_j$$

is given by

$$f_{Z,W}(z,w) =$$

$$\begin{cases} (\Pi k_j) \; _1H \begin{matrix} \Sigma M_{1j}, \Sigma N_{1j}, \Sigma M_{2j}, \Sigma N_{2j}, \Sigma M_{3j}, \Sigma N_{3j} \\ \Sigma P_{1j}, \Sigma Q_{1j}, \Sigma P_{2j}, \Sigma Q_{2j}, \Sigma P_{3j}, \Sigma Q_{3j} \end{matrix} \begin{bmatrix} & \eta_1 & \\ (\Pi g_{1j})z & \eta_2 \; ; \; \eta_3 \\ (\Pi g_{2j})w & \eta_4 \\ & \eta_5 \; ; \; \eta_6 \end{bmatrix} & z,w>0 \\ \\ 0 \quad \text{otherwise} \end{cases}$$

$$(5.14)$$

where the Σ's and Π's are indexed over j and go from 1 to n and

$$\eta_1 = (e_{ij}, E_{ij}), \; i=1 \ldots M_{3j}, \; j=1 \ldots n, \; (e_{ij}, E_{ij}), \; i=M_{3j}+1 \ldots Q_{3j}, \; j=1 \ldots n$$

$$\eta_2 = (a_{ij}, A_{ij}), \; i=1 \ldots N_{1j}, \; j=1 \ldots n, \; (a_{ij}, A_{ij}), \; i=N_{1j}+1 \ldots P_{1j}, \; j=1 \ldots n$$

$$\eta_3 = (c_{ij}, C_{ij}), \; i=1 \ldots N_{2j}, \; j=1 \ldots n, \; (c_{ij}, C_{ij}), \; i=N_{2j}+1 \ldots P_{2j}, \; j=1 \ldots n$$

$$\eta_4 = (f_{ij}, F_{ij}), \; i=1 \ldots N_{3j}, \; j=1 \ldots n, \; (f_{ij}, F_{ij}), \; i=N_{3j}+1 \ldots P_{3j}, \; j=1 \ldots n$$

$$\eta_5 = (b_{ij}, B_{ij}), \; i=1 \ldots M_{1j}, \; j=1 \ldots n, \; (b_{ij}, B_{ij}), \; i=M_{1j}+1 \ldots Q_{1j}, \; j=1 \ldots n$$

$$\eta_6 = (d_{ij}, D_{ij}), \; i=1 \ldots M_{2j}, \; j=1 \ldots n, \; (d_{ij}, D_{ij}), \; i=M_{2j}+1 \ldots Q_{2j}, \; j=1 \ldots n$$

Proof of Theorem 5.3:   From Theorem 2.5,

$$f_{Z,W}(z,w) = M_2^{-1} \left[\prod_{j=1}^{n} M_{s_1,s_2} \{ f_j(x_j, y_j) \} \right] \quad z,w > 0$$

and from the Mellin transform of the bivariate H-function, (3.19), the bivariate density of Z and W is then given by

$$f_{Z,W}(z,w) = M_2^{-1} \left[\prod_{j=1}^{n} \{ k_j g_{1j}^{-s_1} g_{2j}^{-s_2} x_{1j}(s_1) x_{2j}(s_2) x_{3j}(s_1+s_2) \} \right]$$

$$= \left(\prod_{j=1}^{n} k_j \right) M_2^{-1} \left[\prod_{j=1}^{n} \{ x_{1j}(s_1) x_{2j}(s_2) x_{3j}(s_1+s_2) g_{1j}^{-s_1} g_{2j}^{-s_2} \} \right]$$

where $x_{1j}(s_1)$, $x_{2j}(s_2)$, and $x_{3j}(s_1+s_2)$ are defined by (3.7), (3.8), and (3.9) respectively. Recombining like product terms and using (3.6), equation (5.14) may be written directly.

For the case where $M_{ij}=N_{ij}=P_{ij}=Q_{ij}=0$, $i=2,3$, (or for $i=1,3$),

$j=1,\ldots,n$, the bivariate H-functions reduce to univariate H-functions and (5.14) reduces to Carter's Theorem 4-1, (3:52), for the distribution of products of independent H-function variates.

5.3.1.2 The Distribution of Quotients

__Theorem 5.4:__ If X_1,Y_1; X_2,Y_2 are two pairwise independent random variables with bivariate probability density functions $f_1(x_1,y_1)$ and $f_2(x_2,y_2)$ respectively, where

$$f_j(x_j,y_j) = \begin{cases} k_j \; _1H[g_{1j}x_j,g_{2j}y_j] \; , & x_j,y_j > 0 \\ \\ 0 \; , & \text{otherwise} \end{cases}$$

for $j=1,2$, then the bivariate probability density function of the random variables

$$Z = X_1/X_2 \quad ; \quad W = Y_1/Y_2$$

is given by

$$f_{Z,W}(z,w) = \begin{cases} \dfrac{k_1 k_2}{g_{12}^2 g_{22}^2} \; _1H^{M_1,N_1,M_2,N_2,M_3,N_3}_{P_1,Q_1,P_2,Q_2,P_3,Q_3} \left[\begin{array}{cc|c} \dfrac{g_{11}}{g_{12}} z & & n_1 \\ & & n_2 \; ; \; n_3 \\ \dfrac{g_{21}}{g_{22}} w & & n_4 \\ & & n_5 \; ; \; n_6 \end{array} \right] & z,w>0 \\ \\ 0 & \text{otherwise} \end{cases}$$

(5.15)

where $M_i = M_{i1} + N_{i2}$, $N_i = N_{i1} + M_{i2}$, $P_i = P_{i1} + Q_{i2}$, and $Q_i = Q_{i1} + P_{i2}$, for $i=1,2,3$, and

$$\eta_1 = (e_{i1}, E_{i1}), \; i=1\ldots M_{31}, \; (1-f_{12}-4F_{12}, F_{12}), \; i=1\ldots N_{32}$$

$$(e_{i1}, E_{i1}), \; i=M_{31}+1\ldots Q_{31}, \; (1-f_{12}-4F_{12}, F_{12}), \; i=N_{32}+1\ldots P_{32}$$

$$\eta_2 = (a_{i1}, A_{i1}), \; i=1\ldots N_{11}, \; (1-b_{12}-2B_{12}, B_{12}), \; i=1\ldots M_{12},$$

$$(a_{i1}, A_{i1}), \; i=N_{11}+1\ldots P_{11}, \; (1-b_{12}-2B_{12}, B_{12}), \; i=M_{12}+1\ldots Q_{12}$$

$$\eta_3 = (c_{i1}, C_{i1}), \; i=1\ldots N_{21}, \; (1-d_{12}-2D_{12}, D_{12}), \; i=1\ldots M_{22}$$

$$(c_{i1}, C_{i1}), \; i=N_{21}+1\ldots P_{21}, \; (1-d_{12}-2D_{12}, D_{12}), \; i=M_{22}+1\ldots Q_{22}$$

$$\eta_4 = (f_{i1}, F_{i1}), \; i=1\ldots N_{31}, \; (1-e_{12}-4E_{12}, E_{12}), \; i=1\ldots M_{32},$$

$$(f_{i1}, F_{i1}), \; i=N_{31}+1\ldots P_{31}, \; (1-e_{12}-4E_{12}, E_{12}), \; i=M_{32}+1\ldots Q_{32}$$

$$\eta_5 = (b_{i1}, B_{i1}), \; i=1\ldots M_{11}, \; (1-a_{12}-2A_{12}, A_{12}), \; i=1\ldots N_{12},$$

$$(b_{i1}, B_{i1}), \; i=M_{11}+1\ldots Q_{11}, \; (1-a_{12}-2A_{12}, A_{12}), \; i=N_{12}+1\ldots P_{12}$$

$$\eta_6 = (d_{i1}, D_{i1}), \; i=1\ldots M_{21}, \; (1-c_{12}-2C_{12}, C_{12}), \; i=1\ldots N_{22},$$

$$(d_{i1}, D_{i1}), \; i=M_{21}+1\ldots Q_{21}, \; (1-c_{12}-2C_{12}, C_{12}), \; i=N_{22}+1\ldots P_{22}$$

<u>Proof</u> <u>of</u> <u>Theorem</u> <u>5.4</u>: From Theorem 2.5, $n=2$, $a_1=b_1=1$, and $a_2=b_2=-1$, or from (2.22)

$$f_{Z,W}(z,w) = M_2^{-1} [\; M_{s_1,s_2}\{ \; f_1(x_1,y_1) \; \} \; M_{2-s_1,2-s_2}\{ \; f_2(x_2,y_2) \; \} \;]$$

and from the Mellin transform of the bivariate H-function, (3.19), the bivariate density of Z and W is then given by

$$f_{Z,W}(z,w) = M_2^{-1} [\; k_1 g_{11}^{-s_1} g_{21}^{-s_2} \; x_{11}(s_1) x_{21}(s_2) x_{31}(s_1+s_2)$$

$$X \quad k_2 g_{12}^{s_1-2} g_{22}^{s_2-2} \; x_{12}(2-s_1) x_{22}(2-s_2) x_{32}(4-s_1-s_2) \;]$$

$$= \frac{k_1 k_2}{g_{12}^2 g_{22}^2} M_2^{-1} [\ x_{11}(s_1) x_{21}(s_2) x_{31}(s_1+s_2)$$

$$X \quad x_{12}(2-s_1) x_{22}(2-s_2) x_{32}(4-s_1-s_2)(g_{11}/g_{12})^{-s_1}(g_{21}/g_{22})^{-s_2}\ ]$$

where $x_{1j}(\cdot)$, $x_{2j}(\cdot)$, and $x_{3j}(\cdot)$, $j=1,2$, are defined by (3.7), (3.8), and (3.9) respectively. Recombining like product terms and using (3.6), equation (5.15) is obtained.

For the case where $M_{ij}=N_{ij}=P_{ij}=Q_{ij}=0$, $i=2,3$, $j=1,2$, then the bivariate H-functions reduce to univariate H-functions, and (5.15) reduces to Carter's Theorem 4-8, (3:62), for quotients of independent H-function variates.

5.3.1.3 The Distribution of Powers

Because of the restriction in the bivariate H-function definition that s_1 and s_2 must have identical coefficients in the $x_3(s_1+s_2)$ term, a general H-function of the form $_1H[z,w]$ for powers of H-function variates can not be obtained. However, an H-function of the form $_1H[z^{1/p},w^{1/q}]$, where p,q are rational powers of x and y respectively, is possible. For the special case where p=q an H-function of the form $_1H[z,w]$ is possible.

Theorem 5.5: If X,Y are random variables with bivariate probability density function $f_{X,Y}(x,y)$, where

$$f_{X,Y}(x,y) = \begin{cases} k\ _1H[g_1x,g_2y]\ , & x,y > 0 \\ \\ 0\ , \text{otherwise} \end{cases}$$

then the bivariate probability density function of the random variables

$$Z = X^p\ ;\quad W = Y^q$$

p,q rational and $p,q > 0$, is given by

$f_{Z,W}(z,w) =$

$$\begin{cases} K_1H\begin{matrix} M_1,N_1,M_2,N_2,M_3,N_3 \\ P_1,Q_1,P_2,Q_2,P_3,Q_3 \end{matrix}\left[\begin{matrix} g_1z^{1/p} \\ g_2w^{1/q} \end{matrix} \middle| \begin{matrix} (e_i+(2-q-p)E_i,E_i) \\ (a_i+A_i-pA_i,A_i);(c_i+C_i-qC_i,C_i) \\ (f_i+(2-q-p)F_i,F_i) \\ (b_i+B_i-pB_i,B_i);(d_i+D_i-qD_i,D_i) \end{matrix} \right] & z,w>0 \\ \\ 0 \qquad \text{otherwise} \end{cases}$$

(5.16)

where $K = kg_1^{p-1}g_2^{q-1}/pq$

<u>Proof of Theorem 5.5</u>: From Theorem 2.3, letting $t_1=s_1$, $t_2=s_2$,

$$f_{Z,W}(z,w) = M_2^{-1}[\ M_{pt_1-p+1,qt_2-q+1}\{\ f_{X,Y}(x,y)\ \}\ ]$$

$$= M_2^{-1}[\ M_{pt_1-p+1,qt_2-q+1}\{\ k_1H[g_1x,g_2y]\ \}\ ]$$

and from equation (3.19)

$$f_{Z,W}(z,w) = M_2^{-1}\left[kg_1^{-u}g_2^{-v}x_1(u)x_2(v)x_3(u+v)\ \Big|_{\substack{u=pt_1-p+1 \\ v=qt_2-q+1}}\right]$$

$$f_{Z,W}(z,w) = K\ \frac{1}{(2\pi i)^2}\iint x_1(u)x_2(v)x_3(u+v)\left(g_1^p z\right)^{-t_1}\left(g_2^q w\right)^{-t_2}dt_1 dt_2\ \Big|_{\substack{u=pt_1-p+1 \\ v=qt_2-q+1}}$$

where $K = kg_1^{p-1}g_2^{q-1}$ and $\iint$ is the double contour integral for the inverse Mellin transform as defined by (2.6). Let $s_1 = pt_1$, $s_2 = qt_2$, and $ds_1 ds_2 = pq\,dt_1 dt_2$, then

$$f_{Z,W}(z,w) = \frac{K}{pq}\ \frac{1}{(2\pi i)^2}\iint x_1(u)x_2(v)x_3(u+v)\left(g_1^p z\right)^{-s_1/p}\left(g_2^q w\right)^{-s_2/q}ds_1 ds_2\ \Big|_{\substack{u=s_1-p+1 \\ v=s_2-q+1}}$$

$$= \frac{K}{pq}\ \frac{1}{(2\pi i)^2}\iint x_1(u)x_2(v)x_3(u+v)\left(g_1 z^{1/p}\right)^{-s_1}\left(g_2 w^{1/q}\right)^{-s_2}ds_1 ds_2\ \Big|_{\substack{u=s_1-p+1 \\ v=s_2-q+1}}$$

from which (5.16) follows.

For the case where $p=q$, then from (3.15), the H-function in equation (5.16) becomes

$$K_1 H^{M_1,N_1,M_2,N_2,M_3,N_3}_{P_1,Q_1,P_2,Q_2,P_3,Q_3}\left[\begin{array}{c} g_1^p z \\ g_2^p w \end{array}\ \left|\begin{array}{c} (e_1+2E_1-2pE_1,\,pE_1) \\ (a_1+A_1-pA_1,\,pA_1);(c_1+C_1-pC_1,\,pC_1) \\ (f_1+2F_1-2pF_1,\,pF_1) \\ (b_1+B_1-pB_1,\,pB_1);(d_1+D_1-pD_1,\,pD_1) \end{array}\right.\right] \quad z,w>0$$

where $K = k(g_1 g_2)^{p-1}$ $\qquad\qquad\qquad\qquad$ (5.17)

If (2.23) is applied to (5.17) above, the distribution of $Z'=(ZW)^p$ may be found. It can be shown that such a procedure will result in a solution identical to that given by Theorem 5.1, case I, for p=q.

<u>Example</u> <u>5.5</u>: Consider the Kellogg-Barnes I distribution given by

$$f_{X,Y}(x,y) = \frac{4\alpha^{\beta+1}}{\pi\Gamma(\beta+1)} (x^2 + y^2)^\beta e^{-\alpha(x^2 + y^2)} \qquad\qquad x,y > 0$$

$$= \frac{\alpha}{\pi\Gamma(\beta+1)} \, {}_1H^{1,0,1,0,1,0}_{0,1,0,1,1,1} \left[\sqrt{\alpha}\, x \, \middle| \, \begin{array}{c} (\beta,1/2) \\ \text{-----} \; ; \; \text{-----} \\ \sqrt{\alpha}\, y \, \middle| \, (0,1/2) \\ (0,1/2) \; ; \; (0,1/2) \end{array} \right]$$

Now suppose the bivariate probability density function of $Z = X_1/X_2$, $W = Y_1/Y_2$ is desired where X_1, Y_1 and X_2, Y_2 are distributed according to the Kellogg-Barnes I distribution given above. Suppose further that $f_1(x_1,y_1)$ has parameters $\alpha_1 = 1$, $\beta_1 = 0$, and $f_2(x_2,y_2)$ has parameters $\alpha_2 = 1$, $\beta_2 = \beta$. From Theorem 5.4, the bivariate probability density function of Z, W is given by

$$= \frac{1}{\pi^2 \Gamma(\beta+1)} \, {}_1H^{1,1,1,1,0,1}_{1,1,1,1,1,1} \left[\begin{array}{c|c} z & \begin{array}{c} (-1,1/2) \\ (0,1/2) \; ; \; (0,1/2) \\ \end{array} \\ w & \begin{array}{c} (-1-\beta,1/2) \\ (0,1/2) \; ; \; (0,1/2) \end{array} \end{array} \right] \qquad (5.18)$$

from property (3.15), $k = 2$, Equation (5.18) above may be written as

$$= \frac{4}{\pi^2 \Gamma(\beta+1)} \, {}_1H^{1,1,1,1,0,1}_{1,1,1,1,1,1} \left[\begin{array}{c|c} z^2 & \begin{array}{c} (-1,1) \\ (0,1) \; ; \; (0,1) \\ \end{array} \\ w^2 & \begin{array}{c} (-1-\beta,1) \\ (0,1) \; ; \; (0,1) \end{array} \end{array} \right] \qquad (5.19)$$

Recognizing that (5.19) is the H-function form for one of Appell's

hypergeometric functions, special case (3.22), equation (5.19) may be

written as

$$f_{z,W}(z,w) = \frac{4}{\pi^2 \Gamma(\beta+1)} \, \Gamma(\beta+2)\Gamma(1)\Gamma(1)F_1(2+\beta,1,1;2;-z^2,-w^2)$$

$$= \frac{4(\beta+1)}{\pi^2} F_1(2+\beta,1,1;2;-z^2,-w^2)$$

where F_1 is Appell's hypergeometric function of two variables as

defined in Appendix B. The results obtained using the bivariate

H-function and Theorem 5.4 agree with results obtained using Mellin

transform techniques as outlined by Fox (10) and illustrated in

Example 2.3.

Example 5.6: Consider Kellogg-Barnes II distribution given by

$$f_{X,Y}(x,y) = \beta\alpha^2 \, e^{-(\alpha x + \beta y/x)} \qquad x,y > 0$$

$$= \beta\alpha^2 \,\, {}_1H \begin{array}{c} 0,0,1,0,1,0 \\ 0,0,0,1,0,1 \end{array} \left[\begin{array}{c|c} \alpha x & \begin{array}{c} (0,1) \\ \text{-----} \; ; \; \text{-----} \\ \alpha\beta y & \text{-----} \\ \text{-----} \; ; \; (0,1) \end{array} \end{array} \right] \tag{5.20}$$

From (5.17), the bivariate probability density function of $Z=X^{1/2}$, $W=Y^{1/2}$ is given as

$$f_{Z,W}(z,w) = \beta^{1/2}\alpha \,\, {}_1H \begin{array}{c} 0,0,1,0,1,0 \\ 0,0,0,1,0,1 \end{array} \left[\begin{array}{c|c} \sqrt{\alpha}\, z & \begin{array}{c} (1,1/2) \\ \text{-----} \; ; \; \text{-----} \\ \sqrt{\alpha\beta}\, w & \text{-----} \\ \text{-----} \; ; \; (1/2,1/2) \end{array} \end{array} \right] \tag{5.21}$$

From (3.15), for k=2, (5.21) can represented as

$$f_{Z,W}(z,w) = 4\beta^{1/2}\alpha \,\, {}_1H \begin{array}{c} 0,0,1,0,1,0 \\ 0,0,0,1,0,1 \end{array} \left[\begin{array}{c|c} \alpha z^2 & \begin{array}{c} (1,1) \\ \text{-----} \; ; \; \text{-----} \\ \alpha\beta w^2 & \text{-----} \\ \text{-----} \; ; \; (1/2,1) \end{array} \end{array} \right] \tag{5.22}$$

Equation (5.22) above may also be obtained directly from Theorem 5.5, p=q=1/2. Using H-function property (3.16), m=n=1/2, (5.22) may now be expressed as

$$f_{Z,W}(z,w) = 4\beta\alpha^2 zw \; {}_1H\!\begin{array}{l} 0,0,1,0,1,0 \\ 0,0,0,1,0,1 \end{array}\!\left[\begin{array}{c|c} \alpha z^2 & \begin{array}{c} (0,1) \\ \text{-----} \; ; \; \text{-----} \end{array} \\ \alpha\beta w^2 & \begin{array}{c} \text{-----} \\ \text{-----} \; ; \; (0,1) \end{array} \end{array}\right] \qquad (5.23)$$

Comparing the H-function in (5.23) with the H-function in (5.20), the distribution for Z, W may be written as

$$f_{Z,W}(z,w) = 4\beta\alpha^2 \; zw \; \left[\; e^{-(\alpha u + \beta v/u)} \; \right]\Bigg|_{\substack{u=z^2 \\ v=w^2}}$$

$$= 4\beta\alpha^2 \; zw \; e^{-(\alpha z^2 + \beta(w/z)^2)}$$

The solution obtained above may also be found using Theorem 1.4. Let $Z = X^{1/2}$ and $W = Y^{1/2}$, then $X = Z^2$, $Y = W^2$, and the Jacobian is

$$J = \begin{vmatrix} \partial x/\partial z & \partial x/\partial w \\ \partial y/\partial z & \partial y/\partial w \end{vmatrix} = \begin{vmatrix} 2z & 0 \\ 0 & 2w \end{vmatrix} = 4zw$$

From Theorem 1.4,

$$f_{Z,W}(z,w) = f_{X,Y}(z^2,w^2)\,\big|\,J\,\big|$$

$$= 4\beta\alpha^2 \; zw \; e^{-(\alpha z^2 + \beta(w/z)^2)}$$

5.3.2 Transformations for $_2H[x,y]$ Variates

This section presents theorems for transformations of $_2H[x,y]$ variates. Examples for testing the theorems in this section are difficult to derive, and unlike the last section cannot be checked by comparing the results given here to those obtained for independent H-function variates by previous authors. This does not mean that the following theoretical results have no practical applications. Rather, the reverse may be true — this may be the only practical means for obtaining the distribution for transformations of the type given in this section.

5.3.2.1 The Distribution of Products

Theorem 5.6: If X_1,Y_1; X_2,Y_2; $\ldots$; X_n,Y_n are n pairwise independent random variables with bivariate probability density functions $f_1(x_1,y_1)$, $f_2(x_2,y_2)$, $\ldots$, $f_n(x_n,y_n)$ respectively, where

$$f_j(x_j,y_j) = \begin{cases} k_j \; _2H[g_{1j}x_j, g_{2j}y_j] \; , & x_j,y_j > 0 \\ \\ 0 \; , & \text{otherwise} \end{cases}$$

for $j=1,\ldots,n$, then the bivariate probability density function of the random variables

$$Z = \prod_{j=1}^{n} X_j \quad ; \quad W = \prod_{j=1}^{n} Y_j$$

is given by

$$f_{Z,W}(z,w) =$$

$$
\begin{cases}
(\Pi k_j) \; {}_2 H^{\Sigma M_{1j},\Sigma N_{1j},\Sigma M_{2j},\Sigma N_{2j},\Sigma M_{3j},\Sigma N_{3j}}_{\Sigma P_{1j},\Sigma Q_{1j},\Sigma P_{2j},\Sigma Q_{2j},\Sigma P_{3j},\Sigma Q_{3j}} \left[(\Pi g_{1j})z \;\middle|\; (\Pi g_{2j})w \;\middle|\;
\begin{array}{c}
\eta_1 \\
\eta_2 \; ; \; \eta_3 \\
\eta_4 \\
\eta_5 \; ; \; \eta_6
\end{array}
\right] & z,w>0 \\[2em]
0 \quad \text{otherwise} &
\end{cases}
\tag{5.24}
$$

where the Σ's and Π's are indexed over j and go from 1 to n and

$$\eta_1 = (e_{ij}, E_{ij}), \; i=1\ldots M_{3j}, \; j=1\ldots n, \; (e_{ij}, E_{ij}), \; i=M_{3j}+1\ldots Q_{3j}, \; j=1\ldots n$$

$$\eta_2 = (a_{ij}, A_{ij}), \; i=1\ldots N_{1j}, \; j=1\ldots n, \; (a_{ij}, A_{ij}), \; i=N_{1j}+1\ldots P_{1j}, \; j=1\ldots n$$

$$\eta_3 = (c_{ij}, C_{ij}), \; i=1\ldots N_{2j}, \; j=1\ldots n, \; (c_{ij}, C_{ij}), \; i=N_{2j}+1\ldots P_{2j}, \; j=1\ldots n$$

$$\eta_4 = (f_{ij}, F_{ij}), \; i=1\ldots N_{3j}, \; j=1\ldots n, \; (f_{ij}, F_{ij}), \; i=N_{3j}+1\ldots P_{3j}, \; j=1\ldots n$$

$$\eta_5 = (b_{ij}, B_{ij}), \; i=1\ldots M_{1j}, \; j=1\ldots n, \; (b_{ij}, B_{ij}), \; i=M_{1j}+1\ldots Q_{1j}, \; j=1\ldots n$$

$$\eta_6 = (d_{ij}, D_{ij}), \; i=1\ldots M_{2j}, \; j=1\ldots n, \; (d_{ij}, D_{ij}), \; i=M_{2j}+1\ldots Q_{2j}, \; j=1\ldots n$$

Proof of Theorem 5.6: From Theorem 2.5,

$$f_{Z,W}(z,w) = M_2^{-1} \left[\prod_{j=1}^{n} M_{s_1,s_2} \{ f_j(x_j,y_j) \} \right] \quad z,w > 0$$

and from the Mellin transform of ${}_2H[x,y]$, (3.21), the bivariate density of Z and W is then given by

$$f_{Z,W}(z,w) = M_2^{-1} \left[\prod_{j=1}^{n} \{ k_j g_{1j}^{-s_1} g_{2j}^{-s_2} x_{1j}(s_1) x_{2j}(-s_2) x_{3j}(s_1-s_2) \} \right]$$

$$= \left(\prod_{j=1}^{n} k_j \right) M_2^{-1} \left[\prod_{j=1}^{n} \{ x_{1j}(s_1) x_{2j}(-s_2) x_{3j}(s_1-s_2) g_{1j}^{-s_1} g_{2j}^{-s_2} \} \right]$$

where $x_{1j}(s_1)$, $x_{2j}(-s_2)$, and $x_{3j}(s_1-s_2)$ are defined by (3.7), (3.4), and (3.20) respectively. Recombining like product terms and using (3.12), Equation (5.24) may be written directly.

For the case where $M_{ij}=N_{ij}=P_{ij}=Q_{ij}=0$, $i=2,3$, (or for $i=1,3$), $j=1,\ldots,n$, the bivariate H-functions reduce to univariate H-functions and (5.14) reduces to Carter's Theorem 4-1, (3:52), for products of independent H-function variates.

Comparing Theorem 5.6 with Theorem 5.3, it is seen that the form of theorems is identical except that the first deals with $_2H[x,y]$ variates while the second deals with $_1H[x,y]$ variates. It is clear that a more general theorem can be established for both $_1H[x,y]$ variates and $_2H[x,y]$ variates. However, this property does not hold for $_1H[x,y]$ variates and $_2H[x,y]$ variates when transformations of ratios or powers of the variates are concerned.

5.3.2.2 The Distribution of Quotients

Theorem 5.7: If X_1,Y_1; X_2,Y_2 are two pairwise independent random variables with bivariate probability density functions $f_1(x_1,y_1)$ and $f_2(x_2,y_2)$ respectively, where

$$f_j(x_j, y_j) = \begin{cases} k_j\ _2H[g_{1j}x_j, g_{2j}y_j]\ , & x_j, y_j > 0 \\ \\ 0\ , & \text{otherwise} \end{cases}$$

for $j=1,2$, then the bivariate probability density function of the random variables

$$Z = X_1/X_2 \quad ; \quad W = Y_1/Y_2$$

is given by

$$f_{Z,W}(z,w) = \begin{cases} \dfrac{k_1 k_2}{g_{12}{}^2 g_{22}{}^2}\ _2H^{M_1,N_1,M_2,N_2,M_3,N_3}_{P_1,Q_1,P_2,Q_2,P_3,Q_3}\left[\begin{array}{cc|cc} \dfrac{g_{11}}{g_{12}}\ z & & n_1 & \\ & & n_2 \ ; \ n_3 & \\ \dfrac{g_{21}}{g_{22}}\ w & & n_4 & \\ & & n_5 \ ; \ n_6 & \end{array}\right] & z,w>0 \\ \\ 0 & \text{otherwise} \end{cases} \tag{5.25}$$

where $M_i = M_{i1} + N_{i2}$, $N_i = N_{i1} + M_{i2}$, $P_i = P_{i1} + Q_{i2}$, and $Q_i = Q_{i1} + P_{i2}$, for $i=1,2,3$, and

$$n_1 = (e_{i1}, E_{i1}),\ i=1 \ldots M_{31},\ (1-f_{i2}, F_{i2}),\ i=1 \ldots N_{32}$$
$$(e_{i1}, E_{i1}),\ i=M_{31}+1 \ldots Q_{31},\ (1-f_{i2}, F_{i2}),\ i=N_{32}+1 \ldots P_{32}$$

$$n_2 = (a_{i1}, A_{i1}),\ i=1 \ldots N_{11},\ (1-b_{i2}-2B_{i2}, B_{i2}),\ i=1 \ldots M_{12},$$
$$(a_{i1}, A_{i1}),\ i=N_{11}+1 \ldots P_{11},\ (1-b_{i2}-2B_{i2}, B_{i2}),\ i=M_{12}+1 \ldots Q_{12}$$

$$n_3 = (c_{i1}, C_{i1}),\ i=1 \ldots N_{21},\ (1-d_{i2}+2D_{i2}, D_{i2}),\ i=1 \ldots M_{22}$$
$$(c_{i1}, C_{i1}),\ i=N_{21}+1 \ldots P_{21},\ (1-d_{i2}+2D_{i2}, D_{i2}),\ i=M_{22}+1 \ldots Q_{22}$$

$$\eta_4 = (f_{i1}, F_{i1}), \ i=1\ldots N_{31}, \ (1-e_{i2}, E_{i2}), \ i=1\ldots M_{32},$$

$$(f_{i1}, F_{i1}), \ i=N_{31}+1\ldots P_{31}, \ (1-e_{i2}, E_{i2}), \ i=M_{32}+1\ldots Q_{32}$$

$$\eta_5 = (b_{i1}, B_{i1}), \ i=1\ldots M_{11}, \ (1-a_{i2}-2A_{i2}, A_{i2}), \ i=1\ldots N_{12},$$

$$(b_{i1}, B_{i1}), \ i=M_{11}+1\ldots Q_{11}, \ (1-a_{i2}-2A_{i2}, A_{i2}), \ i=N_{12}+1\ldots P_{12}$$

$$\eta_6 = (d_{i1}, D_{i1}), \ i=1\ldots M_{21}, \ (1-c_{i2}+2C_{i2}, C_{i2}), \ i=1\ldots N_{22},$$

$$(d_{i1}, D_{i1}), \ i=M_{21}+1\ldots Q_{21}, \ (1-c_{i2}+2C_{i2}, C_{i2}), \ i=N_{22}+1\ldots P_{22}$$

Proof of Theorem 5.7: From Theorem 2.5, $n=2$, $a_1=b_1=1$, and $a_2=b_2=-1$, or from (2.22)

$$f_{Z,W}(z,w) = M_2^{-1} [\ M_{s_1,s_2} \{ \ f_1(x_1, y_1) \ \} \ M_{2-s_1, 2-s_2} \{ \ f_2(x_2, y_2) \ \} \ ]$$

and fr..m the Mellin transform of $_2H[x,y]$, (3.19), the bivariate density of Z and W is then given by

$$f_{Z,W}(z,w) = M_2^{-1} [\ k_1 g_{11}^{-s_1} g_{21}^{-s_2} \ x_{11}(s_1) x_{21}(-s_2) x_{31}(s_1-s_2)$$

$$X \ k_2 g_{12}^{s_1-2} g_{22}^{s_2-2} \ x_{12}(2-s_1) x_{22}(s_2-2) x_{32}(-s_1+s_2) \ ]$$

$$= \frac{k_1 k_2}{g_{12}^2 g_{22}^2} \ M_2^{-1} [\ x_{11}(s_1) x_{21}(-s_2) x_{31}(s_1-s_2)$$

$$X \ x_{12}(2-s_1) x_{22}(s_2-2) x_{32}(-s_1+s_2)(g_{11}/g_{12})^{-s_1}(g_{21}/g_{22})^{-s_2} \ ]$$

where $x_{1j}(\cdot)$, $x_{2j}(\cdot)$, and $x_{3j}(\cdot)$, $j=1,2$, are defined by (3.7), (3.8),

and (3.9) respectively. Recombining like product terms and using (3.12), Equation (5.25) is obtained.

For the case where $M_{ij}=N_{ij}=P_{ij}=Q_{ij}=0$, $i=2,3$, $j=1,2$, then the bivariate H-functions reduce to univariate H-functions, and (5.25) reduces to Carter's Theorem 4-8, (3:62), for the distribution of quotients of independent H-function variates. Unlike Theorem 5.4, this property does not hold for $i=1,3$. This is due to the fact that for $i=2$, the H-function is defined as y^{s_2} instead of y^{-s_2}.

5.3.2.3 The Distribution of Powers

For the same reasons given in section 5.3.1.3, an H-function of the form $_2H[z,w]$ is not generally obtainable for rational power transformations of the H-function variates X and Y. However, an H-function of the form $_2H[z^{1/p},w^{1/q}]$ is obtainable and for the special case where $p=q$ an H-function of the form $_2H[z,w]$ can be derived.

Theorem 5.8: If X,Y are random variables with bivariate probability density function $f_{X,Y}(x,y)$, where

$$f_{X,Y}(x,y) = \begin{cases} k \ _2H[g_1x,g_2y] \ , & x,y > 0 \\ \\ 0 \ , & \text{otherwise} \end{cases}$$

then the bivariate probability density function of the random variables

$$Z = X^p \quad ; \quad W = Y^q$$

p,q rational and $p,q > 0$, is given by

$$f_{Z,W}(z,w) =$$

$$
\begin{cases}
K_2 H \begin{array}{l} M_1,N_1,M_2,N_2,M_3,N_3 \\ P_1,Q_1,P_2,Q_2,P_3,Q_3 \end{array} \left[\begin{array}{c} g_1 z^{1/p} \\ g_2 w^{1/q} \end{array} \middle| \begin{array}{c} (e_i+(q-p)E_i',E_i) \\ (a_i+A_i-pA_i,A_i);(c_i-C_i+qC_i,C_i) \\ (f_i+(q-p)F_i,F_i) \\ (b_i+B_i-pB_i,B_i);(d_i-D_i+qD_i,D_i) \end{array} \right] & z,w>0 \\ \\ 0 \qquad \text{otherwise} \end{cases}
$$

(5.26)

where $K = k g_1^{p-1} g_2^{q-1}/pq$

<u>Proof of Theorem 5.8</u>: From Theorem 2.3, letting $t_1=s_1$, $t_2=s_2$,

$$f_Z(z) = M_2^{-1}[\; M_{pt_1-p+1,qt_2-q+1}\{\; f_{X,Y}(x,y)\; \}\;]$$

$$= M_2^{-1}[\; M_{pt_1-p+1,qt_2-q+1}\{\; k_2 H[g_1 x, g_2 y]\; \}\;]$$

and from equation (3.21)

$$f_{Z,W}(z,w) = M_2^{-1}\left[k g_1^{-u} g_2^{-v} x_1(u) x_2(-v) x_3(u-v) \middle| \begin{array}{l} u=pt_1-p+1 \\ v=qt_2-q+1 \end{array} \right]$$

$$f_{Z,W}(z,w) = K \frac{1}{(2\pi i)^2} \iint x_1(u) x_2(-v) x_3(u-v) \left| (g_1^p z)^{-t_1} (g_2^q w)^{-t_2} dt_1 dt_2 \right. \begin{array}{l} u=pt_1-p+1 \\ v=qt_2-q+1 \end{array}$$

where $K = k g_1^{p-1} g_2^{q-1}$ and $\iint$ is the double contour integral for the

inverse Mellin transform as defined by (2.6). Let $s_1 = pt_1$, $s_2 = qt_2$, and $ds_1 ds_2 = pq\, dt_1 dt_2$, then

$$f_{Z,W}(z,w) = \frac{K}{pq} \frac{1}{(2\pi i)^2} \iint x_1(u)x_2(-v)x_3(u-v) \left. (g_1{}^p z)^{-s_1/p} (g_2{}^q w)^{-s_2/q} ds_1 ds_2 \right|_{\substack{u = s_1 - p + 1 \\ v = s_2 - q + 1}}$$

$$= \frac{K}{pq} \frac{1}{(2\pi i)^2} \iint x_1(u)x_2(-v)x_3(u-v) \left. (g_1 z^{1/p})^{-s_1} (g_2 w^{1/q})^{-s_2} ds_1 ds_2 \right|_{\substack{u = s_1 - p + 1 \\ v = s_2 - q + 1 \\ u - v = s_1 - s_2 + q - p}}$$

from which (5.26) follows.

For the case where $p = q$, then from (3.15), the H-function in Equation (5.26) becomes

$$K_2 H^{M_1,N_1,M_2,N_2,M_3,N_3}_{P_1,Q_1,P_2,Q_2,P_3,Q_3} \left[\begin{array}{c|c} g_1{}^p z & (e_i, pE_i) \\ & (a_i + A_i - pA_i, pA_i); (c_i - C_i + pC_i, pC_i) \\ g_2{}^p w & (f_i, pF_i) \\ & (b_i + B_i - pB_i, pB_i); (d_i - D_i + pD_i, pD_i) \end{array} \right] \quad z, w > 0$$

where $K = k(g_1 g_2)^{p-1}$ (5.27)

If (2.23) is applied to (5.27), the distribution of $Z' = (ZW)^p$ may be found. It can be shown that such an application will result in a solution identical to that given by Theorem 5.2, case III.

Example 5.7: Consider the Kellogg-Barnes III distribution given by

$$f_{X,Y}(x,y) = \frac{\beta\alpha^c}{\Gamma(c)} \; x^c \; e^{-(\alpha x + \beta xy)} \qquad x,y>0$$

$$= \frac{\beta}{\Gamma(c)} \; 2^H{}^{0,0,0,1,1,0}_{0,0,1,0,0,1} \left[\begin{array}{c|c} & (c,1) \\ \alpha x & \text{-----} \; ; \; (1,1) \\ \frac{\beta}{\alpha} y & \text{-----} \\ & \text{-----} \; ; \; \text{-----} \end{array} \right] \qquad (5.28)$$

From (5.27), the bivariate probability density function of $Z=X^{1/2}$, $W=Y^{1/2}$ is given as

$$f_{Z,W}(z,w) = \frac{\beta^{1/2}}{\Gamma(c)} \; 2^H{}^{0,0,0,1,1,0}_{0,0,1,0,0,1} \left[\begin{array}{c|c} & (c,1/2) \\ \sqrt{\alpha}\, z & \text{-----} \; ; \; (1/2,1/2) \\ \frac{\sqrt{\beta}}{\sqrt{\alpha}} w & \text{-----} \\ & \text{-----} \; ; \; \text{-----} \end{array} \right] \qquad (5.29)$$

From (3.15), k=2, equation (5.29) may be written as

$$f_{Z,W}(z,w) = \frac{4\beta^{1/2}}{\Gamma(c)} \; 2^H{}^{0,0,0,1,1,0}_{0,0,1,0,0,1} \left[\begin{array}{c|c} & (c,1) \\ \alpha z^2 & \text{-----} \; ; \; (1/2,1) \\ \frac{\beta}{\alpha} w^2 & \text{-----} \\ & \text{-----} \; ; \; \text{-----} \end{array} \right] \qquad (5.30)$$

Equation (5.30) may also be obtained by application of Theorem 5.8, $p=q=1/2$. From (3.17), $m=n=1/2$, (5.30) may be written as

$$f_{Z,W}(z,w) = \frac{4\beta}{\Gamma(c)} \; zw \; {}_2H_{0,0,1,0,0,1}^{0,0,0,1,1,0} \left[\begin{array}{cc|c} & & (c,1) \\ \alpha z^2 & & \text{-----} \; ; \; (1,1) \\ \frac{\beta}{\alpha} w^2 & & \text{-----} \\ & & \text{-----} \; ; \; \text{-----} \end{array} \right] \quad (5.31)$$

Comparing (5.31) with (5.28), the bivariate probability density function for Z and W may be given by

$$f_{Z,W}(z,w) = \frac{4\beta}{\Gamma(c)} \; zw \; \left[\; \alpha^c \; u^c \; e^{-(\alpha u + \beta uv)} \; \right]\Big|_{\substack{u=z^2 \\ v=w^2}}$$

$$= \frac{4\beta\alpha^c}{\Gamma(c)} \; wz^{2c+1} \; e^{-(\alpha z^2 + \beta(zw)^2)}$$

The solution obtained above may be obtained through the use of Theorem 1.4. Let $Z=X^{1/2}$ and $W=Y^{1/2}$, then $X=Z^2$, $Y=W^2$, and the Jacobian is

$$J = \begin{vmatrix} \partial x/\partial z & \partial x/\partial w \\ \partial y/\partial z & \partial y/\partial w \end{vmatrix} = \begin{vmatrix} 2z & 0 \\ 0 & 2w \end{vmatrix} = 4zw$$

From Theorem 1.4,

$$f_{Z,W}(z,w) = f_{X,Y}(z^2,w^2) \; |J|$$

$$= \frac{4\beta\alpha^c}{\Gamma(c)} \; wz^{2c+1} \; e^{-(\alpha z^2 + \beta(zw)^2)}$$

5.3.3 The Distribution of a Mix of Product and Quotient

In the previous two sections the bivariate distributions of random variables which were products or quotients of pairwise independent random variables from two or more bivariate distributions. While no theorems are presented, this section discusses the bivariate density functions for random variables that result as a mix of products and quotients of H-function variates.

As an example, suppose the bivariate density function of the dependent random variables Z=X and W=1/Y is desired. From (3.13) it is clear that if X and Y are $_1H[x,y]$ variates, then the resulting bivariate density function is of the form $_2H[x,y]$. Conversely, if X and Y are $_2H[x,y]$ variates, then the resulting density function is of the form $_1H[x,y]$. Theorems for such transformations can be readily deduced, but are not presented here since they are not readily generalized to two or more bivariate H-function distributions as is shown in the following paragraphs.

Now suppose the bivariate density function of the dependent random variables $Z=X_1X_2$ and $W=Y_1/Y_2$, where X_1,Y_1 ; X_2,Y_2 are pairwise independent, is desired. From Theorem 2.5, the bivariate density function of Z and W is given by

$$f_{Z,W}(z,w) = M_2^{-1} \{ M_{s_1,s_2}\{ f_1(x_1,y_1) \} M_{s_1,2-s_2}\{ f_2(x_2,y_2) \} \}$$

If X_1,Y_1 ; X_2,Y_2 are $_1H[x,y]$ variates, then from (3.19) the

distribution of Z and W is given by

$$f_{Z,W}(z,w) = M_2^{-1} \{ k_1 g_{11}^{-s_1} g_{21}^{-s_2} x_{11}(s_1)x_{21}(s_2)x_{31}(s_1+s_2)$$

$$X \quad k_2 g_{12}^{s_1-2} g_{22}^{s_2-2} x_{12}(s_1)x_{22}(2-s_2)x_{32}(2+s_1-s_2) \}$$

Comparing the $x_{31}(s_1+s_2)$ term and the $x_{32}(2+s_1-s_2)$ term, it is clear that it is not possible to make a change of variables substitution and meet the restriction in the $_1H[x,y]$ definition, (3.6), that terms containing both s_1 and s_2 must have s_1 and s_2 be of the same sign. It also fails to meet the restriction in the $_2H[x,y]$ definition, (3.12), that s_1 and s_2 must be of opposite sign for terms containing both s_1 and s_2. The same problem occurs if X_1,Y_1 ; x_2,Y_2 are $_2H[x,y]$ variates. If, however, X_1,Y_1 are $_1H[x,y]$ variates and X_2,Y_2 are $_2H[x,y]$ variates, then the density function of the random variables Z and W is representable as a bivariate H-function and will be of the form $_1H[z,w]$. Conversely, if X_1,Y_1 are $_2H[x,y]$ variates and X_2,Y_2 are $_1H[x,y]$ variates, then the density function of Z and W is also representable as a bivariate H-function and will be of the form $_2H[x,y]$.

To eliminate the problems described above, a more general definition of the bivariate H-function would need to be introduced. Such a definition would be similar to definition (3.6) but would have an added $x_4(s_1-s_2)$ term. Given such a definition, it can be shown that $_1H[x,y]$ and $_2H[x,y]$ would be special cases of this more general

definition. Therefore, Theorems 5.3 and 5.6 would be special cases of a more general theorem for products of H-function variates. Similarly, Theorems 5.4 and 5.7 would just be special cases of a more general theorem for quotients of H-function variates. The primary difficulty with such a definition is the problem of identifying the appropriate contours in the inversion integrals to separate poles in the left and right half planes for the s_1 and s_2 variables. While it would seem that such a definition is viable, additional study on this problem would be required befor the definition could be implemented.

CHAPTER 6

Evaluation of the Bivariate H-function

6.1 General Remarks

In Chapter 5, it is shown that the probability density

function of the product or ratio of two dependent H-function variates

is a univariate H-function. Further, it is shown that the univariate

H-function results from simple algebraic manipulations of the

parameters of the bivariate H-function probability density function.

Then to find the density function of the product or ratio of two

depndent H-function variates, one need merely find the univariate

H-function representation by application of Theorems 5.1 or 5.2. The

computer program given by Cook (5:154-176) may then be applied to give

a numerical evaluation and plot of the univariate H-function density.

For these types of problems, evaluation of the bivariate H-function

density is neither required nor desired.

For problems where the bivariate density function of products

or quotients of pairwise independent variates from two or more

bivariate density functions is desired, evaluation of the bivariate

H-function is required. The procedure for such an evaluation is to

invert the Mellin transform of the bivariate H-function by applying

the residue theorem in an iterative fashion. The procedure, as

outlined in the following sections, is based on the assumption that

any bivariate probability density function of interest is continuous

in each variable throughout its defined region. While such an

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assumption may have some theoretical implications, it provides little or no practical limitation to the evaluation of bivariate H-function densities.

By applying the residue theorem iteratively to find the analytic form of the bivariate probability density function, the procedures outlined by Eldred (7:112-136) and Cook (5:115-118) apply to the bivariate inversion techniques as well. However, the convergence conditions given by Cook (5:61-83) can not immediately be generalized to the bivariate case. While in most instances, one can invert the Mellin transform of the bivariate H-function density by summing residues first in either the left or right half s_2 plane followed by a summation of residues in either the left or right half s_1 plane, a set of bivariate convergence conditions must be established before an analysis of the analytic form of the bivariate H-function density can be completed.

6.2 Complex Analysis in Multiple Dimensions: (91:25-40)

Definition: A function $f(z_1,...,z_k)$ in a domain of its variables is analytic if in some neighborhood of every point $(z_1',...,z_k')$ of the domain it is the sum of an absolutely convergent power series

$$\sum_{n_1=0}^{\infty} ... \sum_{n_k=0}^{\infty} a_{n_1,...,n_k} (z_1-z_1')^{n_1}...(z_k-z_x')^{n_k} \tag{6.1}$$

For $z_j=x_j+iy_j$, $j=1,...,k$, the space of k complex variables,

$z_1, \ldots, z_k$ is the ordinary Euclidean space E_{2k} of the 2k real variables $x_1, y_1, \ldots, x_k, y_k$. If $z' = (z_1', \ldots, z_k')$ is any given point, the neighborhood of this point will be given by the polycylinder

$$C(z', r) : \left| z_j - z_j' \right| < r_j, \quad j = 1, \ldots, k$$

where $r_j > 0$, $j = 1, \ldots, k$

from which

$$C(0, R) : \left| z_j \right| < R_j, \quad j = 1, \ldots, k$$

Theorem 6.1: An analytic function of complex variables is continous and has partial derivatives of all orders which are likewise analytic, and for all series (6.1) we have

$$n_1! \ldots n_k! \, a_{n_1 \ldots n_k} = \frac{\partial^{n_1 + \ldots + n_k} f(z')}{\partial z_1^{n_1} \ldots \partial z_k^{n_k}}$$

Theorem 6.2: If a function $f(z_1, \ldots, z_k)$, all z_i complex, is continuous in domain D, and if in the neighborhood of every point it is analytic in each variable, then $f(z)$ is analytic in D.

Theorem 6.3: If $f(z)$ is analytic in D, all z_i complex, then each expansion (6.1) is unique and is valid in every polycylinder $C(z', R)$ no matter how large, which is contained in D.

The advantage of these theorems is that for a function of complex variables that possesses the property that it is analytic in each variable for all combinations of the other variables, then a

repeated application of the ordinary Cauchy formula leads to the formula

$$f(z_1,\ldots,z_k) = \frac{1}{(2\pi i)^k} \int_{C_1} \frac{d\rho_1}{\rho_1 - z_1} \int_{C_2} \frac{d\rho_2}{\rho_2 - z_2} \ldots \int_{C_k} \frac{f(\rho_1,\ldots,\rho_k)}{\rho_k - z_k} \, d\rho_k \quad (6.2)$$

As in the case of one complex variable, and by Theorem 6.1, since $f(z)$ is analytic in D, equation (6.2) can be differentiated:

$$\frac{\partial^{n_1 + \ldots + n_k} f(z)}{\partial z_1^{n_1} \ldots \partial z_k^{n_k}} = \frac{n_1! \ldots n_k!}{(2\pi i)^k} \int_{C_1} \ldots \int_{C_k} \frac{f(\rho) d\rho_1 \ldots d\rho_k}{(\rho_1 - z_1)^{n_1 + 1} \ldots (\rho_k - z_k)^{n_k + 1}} \quad (6.3)$$

6.3 Inversion of the Bivariate H-function Integral

The bivariate H-function is presented in Chapter 3 in the form of a Mellin transform inversion integral and is given as

$$_1H\{x,y\} = \frac{1}{(2\pi i)^2} \int_{C_1} \int_{C_2} x_1(s_1) x_2(s_2) x_3(s_1 + s_2) x^{-s_1} y^{-s_2} \, ds_1 ds_2 \quad (6.4)$$

where $x_1(s_1)$, $x_2(s_2)$, and $x_3(s_1 + s_2)$ are defined by (3.7), (3.8), and (3.9) respectively. C_1 is a contour in the complex s_1 plane running from $h - i\infty$ to $h + i\infty$ and C_2 is a contour in the complex s_2 plane running from $w - i\infty$ to $w + i\infty$. Both C_1 and C_2 are indented if necessary to avoid the poles of the integrand.

From Theorem 6.2, equation (6.4) may be written as an iterated integral and may be given as

$$_1 H[x,y] = \frac{1}{(2\pi i)^2} \int_{C_1} x_1(s_1) x^{-s_1} \left[\int_{C_2} x_2(s_2) x_3(s_1+s_2) y^{-s_2} ds_2\right] ds_1 \quad (6.5)$$

Consider first the inner integral. From the H-function definition, poles of $\Gamma(d_i+D_i s_2)$, $i=1\dots M_2$, $\Gamma(e_i+E_i(s_1+s_2))$, $i=1\dots M_3$ lie to the left of C_2 and poles of $\Gamma(1-c_i-C_i s_2)$, $i=1\dots N_2$, $\Gamma(1-f_i-F_i(s_1+s_2))$, $i=1\dots N_3$ lie to the right of C_2. Poles of a gamma function occur at nonpositive integer values of its argument. Hence, the poles for the left half s_2 plane may be given by:

$$s_{2ij} = -\frac{d_i+j}{D_i}, \quad j = 0,1,2,\dots$$

and

$$s_{2ij} = -\frac{e_i+E_i s_1+j}{E_i}, \quad j = 0,1,2,\dots$$

for the factors

$$\prod_{i=1}^{M_2} \Gamma(d_i+D_i s_2) \quad \text{and} \quad \prod_{i=1}^{M_3} \Gamma(e_i+E_i(s_1+s_2))$$

respectively. Similarly, poles for the right half s_2 plane may be given by

$$s_{2ij} = \frac{1-c_i+j}{C_i}, \quad j = 0,1,2,\dots$$

and

$$s_{2ij} = \frac{1-f_i-F_i s_1+j}{F_i}, \quad j = 0,1,2,\dots$$

for the factors

$$\prod_{i=1}^{N_2} \Gamma(1-c_i-C_i s_2) \quad \text{and} \quad \prod_{i=1}^{N_3} \Gamma(1-f_i-F_i(s_1+s_2))$$

respectively.

In evaluating the residues of the poles given above, s_1 must be carried as a constant to be treated as a variable in the next integral. Because of this, an analytic form after inversion in the s_2 plane must be presented to the s_1 plane before an analysis of the location of the poles in the s_1 plane may be conducted. However, an analytic form can not be obtained for the general case until a valid convergence proof, similar to that given by Cook, (3), for the univariate H-function, is obtained. However, analysis of the location of the poles in the s_1 plane is possible given a specific set of parameters and will be shown in examples at the end of this section.

While a complete analysis of the location of the poles in the s_1 plane is not possible without a convergence proof, the poles from the $x_1(s_1)$ term may be identified and are given as

$$s_{1ij} = -\frac{b_i+j}{B_i} , \quad j = 0,1,2,\ldots$$

and

$$s_{1ij} = \frac{1-a_i+j}{A_i} , \quad j = 0,1,2,\ldots$$

for the factors

$$\prod_{i=1}^{M_1} \Gamma(b_i + B_i s_2) \quad \text{and} \quad \prod_{i=1}^{N_1} (1 - a_i - A_i s_1)$$

respectively. The top set of poles are valid for the left half plane and the bottom set are valid for the right half plane.

Similar results may be derived for $_2H[x,y]$. $_2H[x,y]$ is presented in Chapter 3 in the form of a Mellin transform inversion integral and is given as

$$_2H[x,y] = \frac{1}{(2\pi i)^2} \int_{C_1} \int_{C_2} x_1(s_1) x_2(-s_2) x_3(s_1-s_2) x^{-s_1} y^{-s_2} ds_1 ds_2 \quad (6.6)$$

where $x_1(s_1)$, $x_2(-s_2)$, and $x_3(s_1-s_2)$ are defined by (3.7), (3.4), and (3.20) respectively. From Theorem 6.2, equation (6.6) may be written as an iterated integral and may be given as

$$_2H[x,y] = \frac{1}{(2\pi i)^2} \int_{C_1} x_1(s_1) x^{-s_1} [\int_{C_2} x_2(-s_2) x_3(s_1-s_2) y^{-s_2} ds_2] ds_1 \quad (6.7)$$

Consider first the inner integral. From the $_2H[x,y]$ definition, poles of $\Gamma(d_i - D_i s_2)$, $i=1\ldots M_2$, $\Gamma(e_i + E_i(s_1-s_2))$, $i=1\ldots M_3$ lie to the right of C_2 and poles of $\Gamma(1-c_i+C_i s_2)$, $i=1\ldots N_2$, $\Gamma(1-f_i-F_i(s_1-s_2))$, $i=1\ldots N_3$ lie to the left of C_2. Since poles of a gamma function occur at nonpositive integer values of its argument, the poles for the left half s_2 plane may be given by:

$$s_{2ij} = -\frac{1-c_i+j}{C_i} , \quad j = 0,1,2,\ldots$$

and

$$s_{2ij} = -\frac{1-f_i-F_i s_1+j}{F_i} , \quad j = 0,1,2,\ldots$$

for the factors

$$\prod_{i=1}^{N_2} \Gamma(1-c_i+C_i s_2) \quad \text{and} \quad \prod_{i=1}^{N_3} \Gamma(1-f_i-F_i(s_1-s_2))$$

respectively. Similarly, the poles for the right half plane are given by

$$s_{2ij} = \frac{d_i+j}{D_i} , \quad j = 0,1,2,\ldots$$

and

$$s_{2ij} = \frac{e_i+E_i s_1+j}{E_i} , \quad j = 0,1,2,\ldots$$

for the factors

$$\prod_{i=1}^{M_2} \Gamma(d_i-D_i s_2) \quad \text{and} \quad \prod_{i=1}^{M_3} \Gamma(e_i+E_i(s_1-s_2))$$

respectively.

Comparing the poles in the s_2 plane for $_2H[x,y]$ to those in the s_2 plane for $_1H[x,y]$, it is clear that for cases where $_1H[x,y]$ and $_2H[x,y]$ have identical parameters the poles for both are exactly identical except that they are in opposite half planes. Poles in the s_1 plane are exactly identical for both $_1H[x,y]$ and $_2H[x,y]$. If s_2 is

replaced by $-s_2$ in (6.7), then the inversion integral may be written as

$$_2H[x,y] = \frac{1}{(2\pi i)^2} \int_{C_1} x_1(s_1)x^{-s_1} [\int_{C_2} x_2(s_2)x_3(s_1+s_2)y^{s_2}ds_2] ds_1 \quad (6.8)$$

From (6.8), the poles in both the s_1 and s_2 planes are identical for $_1H[x,y]$ and $_2H[x,y]$. Therefore, the residues of $_1H[x,y]$ will be identical to those of $_2H[x,y]$ except that $_1H[x,y]$ will have variables (x,y) and $_2H[x,y]$ will have variables $(x,1/y)$. This could also have been derived directly from (3.13). From this analysis, it is clear that if a set of convergence conditions can be derived for $_1H[x,y]$, then it will also be valid for $_2H[x,y]$.

All of the bivariate H-function probability density functions given in section 4.3 were inverted to reproduce their analytic forms using the techniques outlined above. The exception to this is the Kellogg-Barnes I distribution. It can not be inverted for the general case because specific values of the β parameter determine different pole sequences. However, once β is set, the residue sequence may be established and the distribution may be inverted. The Kellogg-Barnes distribution was inverted for values of $\beta = 1$ and $\beta = 2$.

The following examples demonstrate the inversion techniques outlined above. The bivariate gamma and beta distributions are demonstrated in Examples 6.1 and 6.2 because they are the most difficult to invert and because they demonstrate the validity of the inversion process when $f_{X,Y}(x,y)$ is positive for only a portion of the

positive quadrant. Example 6.3 demonstrates the inversion techniques
for $_2H[x,y]$ variates.

<u>Example</u> <u>6.1</u>: Consider McKay's bivariate gamma distribution given by

$$f_{X,Y}(x,y) = \frac{a^{p+q}}{\Gamma(p)\Gamma(q)} x^{p-1} (y-x)^{q-1} e^{-ay} \quad , \quad y>x>0$$

$$= \frac{a^2}{\Gamma(p)} \, _1H_{1,1,0,0,0,1}^{1,0,0,0,1,0} \left[\begin{array}{c|c} ax & (p+q-2,1) \\ & (p+q-1,1) \; ; \; ----- \\ ay & ----- \\ & (p-1,1) \; ; \; ----- \end{array} \right]$$

$$= \frac{a^2}{\Gamma(p)} \frac{1}{(2\pi i)^2} \int\int \frac{\Gamma(p-1+s_1)\Gamma(p+q-2+s_1+s_2)}{\Gamma(p+q-1+s_1)} (ax)^{-s_1}(ay)^{-s_2} ds_1 ds_2$$

$$(6.9)$$

From Theorem 6.2, the Mellin transform inversion integral in (6.9) may
be treated as an iterated integral and may be written as

$$f_{X,Y}(x,y) =$$

$$\frac{a^2}{\Gamma(p)} \frac{1}{2\pi i} \int \frac{\Gamma(p-1+s_1)(ax)^{-s_1}}{\Gamma(p+q-1+s_1)} [\frac{1}{2\pi i} \int \Gamma(p+q-2+s_1+s_2)(ay)^{-s_2} ds_2] \, ds_1$$

$$(6.10)$$

Consider now the inner integral in (6.10). The integral may be

evaluated by summing the residues in the left half s_2 plane. The poles of the integrand are given by

$$s_{2j} = -(p+q-2+s_1+j) \quad , \quad j = 0,1,2,\ldots$$

Summing over j the residues of the poles given by s_{2j} and using the property

$$\Gamma(x) = \frac{\Gamma(x+n)}{(x+n-1)(x+n-2)\ldots x}$$

yields

Σ residues =

$$\sum_{j=0}^{\infty} \frac{\Gamma(p+q-1+j+s_1+s_{2j})(s_1+s_{2j}+p+q-2+j)(y/a)^{-s_{2j}}}{(s_1+s_{2j}+p+q-2)(s_1+s_{2j}+p+q-1)\ldots(s_1+s_{2j}+p+q-2+j)} \Bigg|_{s_{2j}=-(j+p+q-2+s_1)}$$

$$= \sum_{j=0}^{\infty} \frac{(ay)^{s_1+p+q+j-2}(-1)^j}{j!}$$

$$= (ay)^{s_1+p+q-2} \sum_{j=0}^{\infty} \frac{(-1)^j (ay)^j}{j!}$$

Recognizing that the summation is just the power series expansion for e^{-ay}, the sum of the residues is now given by

$$\Sigma \text{ residues} = (ay)^{p+q-2} (ay)^{s_1} e^{-ay}$$

Substituting the solution for the inner integral back into (6.10) gives

$$f_{X,Y}(x,y) = \frac{a^{p+q}}{\Gamma(p)} \, y^{p+q-2} \, e^{-ay} \, [\frac{1}{2\pi i} \int \frac{\Gamma(p-1+s_1)}{\Gamma(p+q-1+s_1)} \, (x/y)^{-s_1} ds_1] \qquad (6.11)$$

The inversion integral inside the [ ]'s may be evaluated by summing the residues in the left half s_1 plane. The poles of the integrand are given by

$$s_{1j} = -(p-1+j) \quad , \quad j = 0,1,2,\ldots$$

Summing over j the residues of the poles given by s_{1j} yields

Σ residues =

$$\sum_{j=0}^{\infty} \frac{\Gamma(p+1+s_{1j}+j)(p-1+s_{1j}+j)(x/y)^{-s_{1j}}}{(p-1+s_{1j})(p+s_{1j})\cdots(p-1+s_{1j}+j)\Gamma(p+q-1+s_{1j})} \quad s_{1j}=1-p-j$$

Using the property $\Gamma(q-j) = \Gamma(q)/(q-1)(q-2)\cdots(q-j)$ and recognizing that

$$\frac{1}{(q-1)(q-2)\cdots(q-j)} = \frac{(q-1)!}{(q-1-j)!}$$

gives

$$\Sigma \text{ residues} = \frac{x^{p-1}y^{1-p}}{\Gamma(q)} \sum_{j=0}^{\infty} \frac{(-1)^j(q-1)!(x/y)^j}{(q-1-j)!\ j!}$$

But, from (94:1), the series above is just the binomial expansion for $(1-x/y)^{q-1}$, where for convergence, $(x/y)^2 < 1$, or restated, $y > x$. Then

$$\Sigma \text{ residues} = \frac{x^{p-1}y^{1-p}}{\Gamma(q)} (1-x/y)^{q-1}$$

$$= \frac{x^{p-1}\ y^{2-p-q}\ (y-x)^{q-1}}{\Gamma(q)}$$

Substituting this identity back into (6.11) yields

$$f_{X,Y}(x,y) = \frac{a^{p+q}}{\Gamma(p)\Gamma(q)}\ x^{p-1}\ (y-x)^{q-1}\ e^{-ay} \quad , \quad y>x>0$$

which is identical to the analytic form given in (6.9).

Example 6.2: Consider the bivariate Beta distribution given by

$$f_{X,Y}(x,y) = \frac{\Gamma(p_1+p_2+p_3)}{\Gamma(p_1)\Gamma(p_2)\Gamma(p_3)}\ x^{p_1-1}\ y^{p_2-1}\ (1-x-y)^{p_3-1}$$

$$= \frac{\Gamma(p_1+p_2+p_3)}{\Gamma(p_1)\Gamma(p_2)}\ {}_1H^{1,0,1,0,0,0}_{0,1,0,1,1,0} \left[\begin{array}{c|c} x & \text{-----} \\ y & \text{-----} \ ; \ \text{-----} \\ & (p_1+p_2+p_3-2,1) \\ & (p_1-1,1) \ ; \ (p_2-1,1) \end{array} \right]$$

$$= \frac{\Gamma(p_1+p_2+p_3)}{\Gamma(p_1)\Gamma(p_2)} \frac{1}{2\pi i} \int \Gamma(s_1+p_1-1)x^{-s_1} \left[\frac{1}{2\pi i}\int \frac{\Gamma(s_2+p_2-1)y^{-s_2}}{\Gamma(s_1+s_2+p_1+p_2+p_3-2)}ds_2\right] ds_1$$

where $x,y>0$ and $x+y\leq 1$.   (6.12)

From Theorem 6.2, (6.12) may be inverted by the residue theorem applied to the s_2 plane, treating s_1 as a constant, followed by inversion in the s_1 plane by residues. Consider the inner inversion integral. The poles of the integrand are given by

$$s_{2j} = -(p_2-1+j) \quad , \quad j = 0,1,2,\ldots$$

from which the sum of the residues may be given by

Σ residues $=$

$$\sum_{j=0}^{\infty} \left.\frac{\Gamma(s_{2j}+p_2+j)(s_{2j}+p_2-1+j)y^{-s_{2j}}}{(s_{2j}+p_2-1)(s_{2j}+p_2)\cdots(s_{2j}+p_2-1+j)\Gamma(s_1+s_{2j}+p_1+p_2+p_3-2)}\right|_{s_{2j}=1-p_2-j}$$

$$= \sum_{j=0}^{\infty} \frac{y^{p_2-1+j}(-1)^j}{j!\ \Gamma(s_1+p_1+p_3-1-j)}$$

$$= y^{p_2-1} \sum_{j=0}^{\infty} \frac{(-y)^j}{j!\ \Gamma(s_1+p_1+p_3-1-j)}$$

Substituting this series back into (6.12) yields

$$f_{X,Y}(x,y) = \frac{\Gamma(p_1+p_2+p_3)}{\Gamma(p_1)\Gamma(p_2)} y^{p_2-1} [\frac{1}{2\pi i}\int \sum_{j=0}^{\infty} \frac{(-y)^j \Gamma(s_1+p_1-1)x^{-s_1}}{j!\Gamma(s_1+p_1+p_3-1-j)} \, ds_1] \qquad (6.13)$$

The inversion integral inside the [ ]'s may be inverted by summing the residues in the left half s_1 plane. From the integrand of the inversion integral, the poles of the integrand by

$$s_{1i} = -(p_1-1+i) \quad , \quad i = 0,1,2,\ldots$$

from which the sum of the residues may be given by

Σ residues =

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-y)^j\Gamma(s_{1i}+p_1)(s_{1i}+p_1-1+i)x^{-s_{1i}}}{j!(s_{1i}+p_1-1)(s_{1i}+p_1)\cdots(s_{1i}+p_1-1+i)\Gamma(s_{1i}+p_1+p_3-1-j)}\Bigg|_{s_{1i}=1-p_1-i}$$

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-y)^j x^{p_1-1+i}(-1)^i}{i!j!\ \Gamma(p_3-i-j)}$$

$$= \frac{x^{p_1-1}}{\Gamma(p_3)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(p_3-1,i+j)(-x)^i(-y)^j}{i!\ j!}$$

$$= \frac{x^{p_1-1}}{\Gamma(p_3)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(1-p_3,i+j)x^i\ y^j}{i!\ j!}$$

where

$$(1-p_3, i+j) = \frac{\Gamma(1-p_3+i+j)}{\Gamma(1-p_3)}$$

From (B.4), the double series above may be represented as an Appell's hypergeometric function of two variables. The sum of the residues may then be given as

$$\Sigma \text{ residues} = \frac{x^{p_1-1}}{\Gamma(p_3)} F_2(1-p_3, b, b'; b, b'; x, y)$$

where the parameters b and b' are arbitrary real positive constants and for convergence $x+y<1$. From Appendix B, special cases, it is seen that

$$F_2(1-p_3, b, b'; b, b'; x, y) = (1-x-y)^{p_3-1}$$

Substituting these values back into (6.13) yields

$$f_{X,Y}(x,y) = \frac{\Gamma(p_1+p_2+p_3)}{\Gamma(p_1)\Gamma(p_2)\Gamma(p_3)} x^{p_1-1} y^{p_2-1} (1-x-y)^{p_3-1} \quad , \quad x+y<1$$

which agrees with the analytic form for the bivariate beta distribution given in (6.12).

<u>Example</u> <u>6.3</u>: The Kellogg-Barnes III distribution is given by

$$f_{X,Y}(x,y) = \frac{\beta\alpha^c}{\Gamma(c)} x^c e^{-(\alpha x+\beta xy)} \quad , \quad \begin{array}{l} x,y>0 \\ \alpha,\beta>0 \ , \ c>2 \end{array}$$

$$= \frac{\beta}{\Gamma(c)} \, 2^H {}^{0,0,0,1,1,0}_{0,0,1,0,0,1} \left[\begin{array}{c|c} \alpha x \\ \frac{\beta}{\alpha} y \end{array} \middle| \begin{array}{c} (c,1) \\ \text{-----} \; ; (1,1) \\ \text{-----} \\ \text{-----} \; ; \text{-----} \end{array} \right]$$

$$= \frac{\beta}{\Gamma(c)} \, \frac{1}{2\pi i} \int \Gamma(-s_2)(\beta y/\alpha)^{s_2} \left[\frac{1}{2\pi i} \int \Gamma(c+s_1+s_2)(\alpha x)^{-s_1} ds_1 \right] ds_2 \qquad (6.14)$$

Inverting first with respect to s_1, the integrand has poles at

$$s_{1j} = -(s_2 + c + j) \quad , \quad j = 0, 1, 2, \ldots$$

Summing over j the residues in the left half s_1 plane given by s_{1j} yields

$$\Sigma \text{ residues} = \sum_{j=0}^{\infty} \frac{\Gamma(c+s_{1j}+s_2+1+j)(c+s_{1j}+s_2+j)(\alpha x)^{-s_{1j}}}{(c+s_{1j}+s_2)(c+s_{1j}+s_2+1)\ldots(c+s_{1j}+s_2+j)} \Bigg|_{s_{1j}=-(s_2+c+j)}$$

$$= \sum_{j=0}^{\infty} \frac{\alpha^{s_2+c+j} \, x^{s_2+c+j} \, (-1)^j}{j!}$$

$$= \alpha^c \, x^c \, (\alpha x)^{s_2} \sum_{j=0}^{\infty} \frac{(\alpha x)^j \, (-1)^j}{j!}$$

$$= \alpha^c \, x^c \, (\alpha x)^{s_2} \, e^{-\alpha x}$$

Substituting this back into (6.14) gives

$$f_{X,Y}(x,y) = \frac{\beta\alpha^c}{\Gamma(c)} \; x^c \; e^{-\alpha x} \; [\frac{1}{2\pi i}\int \Gamma(-s_2)(\beta xy)^{s_2} \; ds_2] \qquad (6.15)$$

The inversion integral inside the [ ]'s may be inverted by summing the residues in the right half s_2 plane. The poles of the integrand are given by

$$s_{2j} = j \quad , \quad j = 0,1,2,\ldots$$

The sum of the residues is then given by

$$\Sigma \text{ residues} = \sum_{j=0}^{\infty} \frac{\Gamma(1+j-s_{2j})(j-s_{2j})(\beta xy)^{s_{2j}}}{(-s_{2j})(-s_{2j}+1)\ldots(-s_{2j}+j)} \Bigg|_{s_{2j}=j}$$

$$= \sum_{j=0}^{\infty} \frac{(\beta xy)^j \; (-1)^j}{j!}$$

$$= e^{-\beta xy}$$

Substituting this back into (6.15) gives

$$f_{X,Y}(x,y) = \frac{\beta\alpha^c}{\Gamma(c)} \; x^c \; e^{-(\alpha x+\beta xy)}$$

This agrees with the analytic form given in (6.14).

CHAPTER 7

Conclusion and Recommendations

The main purpose of this dissertation has been to demonstrate a practical technique for determining the probability density function and the cumulative distribution function of products, quotients, or powers of two dependent H-function variates. This has been accomplished in section 5.2. While trying to accomplish this purpose, other contributions have resulted.

Fox (10) and Subrahamian (19) show how to find the probability density function for a simple product or ratio of two dependent variables or of pairwise independent variables from two bivariate distributions using double Mellin transforms. This work has been extended in section 2.4 to account for arbitrary rational powers of the variables. This section also includes extensions of Fox's work to n sets of pairwise independent variables. To facilitate manipulations of integral transforms, extensions to the univariate Mellin transform properties have been established and are presented in section 2.2.

A second type of H-function which is strongly related to the first, is defined in section 3.2. Associated properties for both H-function types are given in section 3.3. While many applications can incorporate the two definitions into one by simply allowing one of the variables to be inverted in the H-function definition, it is simpler in other applications, such as probability and statistics, to keep the two types of H-functions separated.

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A remarkably rewarding area of study has been in the area of the cumulative distribution function of an H-function probability density function. First, by section 4.4, the improved Laplace transform for the univariate H-function given by Cook (5) has been shown to be unnecessary if the H-function is a probability density function. Specifically, the Laplace transform of the univariate H-function distribution given by Carter (3) is sufficient and complete. This not only leads to a simplification of the form of the cumulative distribution function, but it also provides great insight as to the range of values the parameters of the H-function distribution can undertake. Second, the cumulative distribution function of a bivariate H-function probability density function has been shown to be a bivariate H-function. And, third, the study of the cumulative distribution function has led to a formula for finding the constant for the H-function distribution, given in section 4.5.

As stated in the first paragraph, the main thrust of this dissertation has been to find the density function of a product or ratio of two dependent H-function variates. A natural extension to this type of problem is to find the bivariate density function of two H-function variates which are products or ratios of pairwise independent variates from two or more bivariate H-function distributions. This has been accomplished in section 5.3. However, unlike the univariate H-function combinations, section 5.3 shows that the bivariate density function of some combinations of H-function

variates will not result in a bivariate H-function distribution.

The following recommendations are made for directions of future work on bivariate H-function distributions:

1. A set of convergence conditions similar to those given by Cook (5) for the univariate case are needed for the bivariate H-function. While several results have been obtained on the bivariate H-function distribution, further significant advancements would be difficult without such a set of conditions.

2. Use of the double Laplace transform to find the probability density function of the sum of two dependent random variables is another area of possible research. Sneddon (106) devotes a section to the double Laplace transform and provides some insight to the solution of this type problem.

3. An extension of Prasad's theorems (57) to the bivariate case would allow a researcher to develop a formula for the Laplace transform of the bivariate H-function. Such a formula could prove useful in the study of sums of dependent H-function variates.

4. In Chapter 6, the analytic form of the bivariate H-function is analyzed by performing the contour integration iteratively. While this works well for special cases, it is difficult to utilize this procedure to write a general program to invert the bivariate H-function. It is believed that if the location of the two dimensional poles can be systematically analysed, a more efficient inversion technique can be accomplished by using the properties of the

multiple Laurent's series given in (6.1).

    5. Given a set of convergence conditions, a study of the various methods of numerical inversion of the H-function should be considered a must. This work could include a study of the univariate computational efficiency as well as a generalized code to numerically invert the bivariate H-function.

    6. Another possible area is the use of H-functions to study probability density functions defined over the entire real line. This area of research may be applied to univariate as well as the bivariate distributions. The positive-negative component methods developed by Epstein (9), Springer and Thompson (18), and Fox (10) should accommodate such usage, particularly for distributions symmetric about zero.

    7. The application of bivariate H-functions to the fitting of contours to correlated data is another unexplored realm. Being the most general of the special functions of two variables, the bivariate H-function appears to be as suitable for contour-fitting as it has been for analyzing probability density functions of products and ratios of dependent random variables.

APPENDIX A

Classical Bivariate Probability

Distributions

APPENDIX A: <u>Classical Bivariate Probability Distributions</u>

<u>Normal</u>

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right\}$$

$$-\infty < x,y < \infty, \quad \rho \neq 1,-1$$

<u>Uniform</u>

I) Morgensterm

$$f_{X,Y}(x,y) = 1 + \rho(2x - 1)(2y - 1)$$

$$0 \leq x,y \leq 1$$

II) Plackett

$$f_{X,Y}(x,y) = \frac{\rho\{(\rho - 1)(x + y - 2xy) + 1\}}{[\{1 + (\rho - 1)(x + y)\}^2 - 4\rho(\rho - 1)xy]^{3/2}}$$

$$0 \leq x,y \leq 1, \quad \rho \neq 1$$

<u>Cauchy</u>

$$f_{X,Y}(x,y) = \frac{\rho}{2\pi}[\rho^2 + x^2 + y^2]^{-3/2}$$

$$-\infty < x,y < \infty, \quad \rho > 0$$

Beta

$$f_{X,Y}(x,y) = \frac{\Gamma(p_1 + p_2 + p_3)}{\Gamma(p_1)\Gamma(p_2)\Gamma(p_3)} \; x^{p_1-1} \; y^{p_2-1} \; (1 - x - y)^{p_3-1}$$

$$x,y \geq 0, \quad x + y \leq 1$$

Gamma

I) McKay

$$f_{X,Y}(x,y) = \frac{a^{p+q}}{\Gamma(p)\Gamma(q)} \; x^{p-1} \; (y - x)^{q-1} \; e^{-ay}$$

$$y > x > 0$$

II) Cherian

$$f_{X,Y}(x,y) = \frac{e^{-(x-y)}}{\Gamma(p_1)\Gamma(p_2)\Gamma(p_3)} \int_0^{\min(x,y)} z^{p_3-1} (x-z)^{p_1-1} (y-z)^{p_2-1} e^z dz$$

$$x,y \geq 0$$

III) Wicksell - Kibble

$$f_{X,Y}(x,y) = \frac{1}{c^{p-1}\Gamma(p)} \; e^{-(x + y)/(1 - c)} \sum_{k=0}^{\infty} \frac{(cxy)^{p+k-1}}{k!\Gamma(p + k)(1 - c)^{p+2k}}$$

$$x,y \geq 0$$

Exponential

I) Gumbel

$$f_{X,Y}(x,y) = \{(1 + ax)(1 + ay) - a\} \exp(-x - y - axy)$$

$$x,y > 0$$

II) Marshall - Olkin

$$F_{X,Y}(x,y) = \exp\{-\lambda_1 x - \lambda_2 y + \lambda_3 \max(x,y)\}$$

$$x,y > 0$$

Pareto

$$f_{X,Y}(x,y) = \frac{p(p + 1)(ab)^{p+1}}{(bx + ay - ab)^{p+2}}$$

$$x > a, \quad y > b$$

Student's t

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \frac{f}{(f-2)} \left\{ 1 + \frac{1}{(f-2)(1-\rho^2)}(x^2 - 2\rho xy + y^2) \right\}^{-(f+2)/2}$$

$$-\infty < x,y < \infty$$

F-distribution

$$f_{X,Y}(x,y) = \Gamma(v/2)v_0^{-v/2} \prod_{i=0}^{2} \frac{v^{v_1/2}}{\Gamma(v_1/2)} \frac{x^{(v_1/2)-1} y^{(v_2/2)-1}}{[(1 + v_1 x + v_2 y)/v_0]^{v/2}}$$

$$x,y > 0, \quad v = v_0 + v_1 + v_2$$

APPENDIX B

Hypergeometric Functions

APPENDIX B: Hypergeometric Functions

Appel's Functions: (21:281-302;7:13-36)

The hypergeometric series of one variable is given as

$$_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)x^n}{(c,n)n!} \qquad (B.1)$$

The symbol (u,k), where u denotes any number real or complex, and k is any real integer, is defined by

$$(u,k) = \frac{\Gamma(u+k)}{\Gamma(u)} = u(u+1)\ldots(u+k-1), \quad k \geq 0$$

and

$$(u,-k) = \frac{(-1)^k}{(1-u,k)}, \quad k < 0$$

Elements a, b, and c are the parameters of the series and x is the variable of the series. The series is not defined if c is a non-positive integer, unless a or b is also a negative integer such that $-c < -a$ or $-c < -b$.

Appell derived an expression for a hypergeometric function of two variables by considering the simple product of two Gauss functions.

$$_2F_1(a,b;c;x)\,_2F_1(a',b';c';y)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a,m)(a',n)(b,m)(b',n)x^m y^n}{(c,m)(c',n)m!n!} \tag{B.2}$$

and replacing, in turn, each pair of products $(a,m)(a',n)$, for example, by the composite product $(a,m+n)$. By considering the possible composite product combinations, Appell derived the following four hypergeometric functions of two variables.

$$F_1(a,b,b';c;x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a,m+n)(b,m)(b',n)x^m y^n}{(c,m+n)m!n!} \tag{B.3}$$

where for convergence, $|x| < 1$, $|y| < 1$

$$F_2(a,b,b';c,c';x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a,m+n)(b,m)(b',n)x^m y^n}{(c,m)(c',n)m!n!} \tag{B.4}$$

where for convergence, $|x| + |y| < 1$

$$F_3(a,a',b,b';c;x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a,m)(a',n)(b,m)(b',n)x^m y^n}{(c,m+n)m!n!} \tag{B.5}$$

where for convergence, $|x| < 1$, $|y| < 1$

$$F_4(a,b;c,c';x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a,m+n)(b,m+n)x^m y^n}{(c,m)(c',n)m!n!} \tag{B.6}$$

where for convergence, $|x|^{1/2} + |y|^{1/2} < 1$

Special Cases: (12:160)

$$F_1(a,b,b';a;x,y) = (1 - x)^{-b}(1 - y)^{-b'}$$

$$F_2(a,b,b;b,b';x,y) = (1 - x - y)^{-a}$$

$$F_2(a,b,b';a,b';x,y) = (1 - x - y)^{-b}(1 - y)^{b-a}$$

$$F_2(a,b,b';b,a;x,y) = (1 - x)^{b'-a}(1 - x - y)^{-b'}$$

Kampé de Feriét's Function: (7:29-33)

Appells functions may be represented as special cases of a more generalized double hypergeometric function of higher order. This function was first defined by Kampé de Feriét in 1921 and is given by

$$F\begin{bmatrix} A & a_1;\ldots a_A \\ B & b_1,b_1';\ldots b_B,b_B' \\ C & c_1;\ldots c_C \\ D & d_1,d_1';\ldots d_D,d_D' \end{bmatrix} x,y$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\prod\limits_{i=1}^{A}(a_i,m+n)\prod\limits_{i=1}^{B}(b_i,m)(b_1',n)x^m y^n}{\prod\limits_{j=1}^{C}(c_i,m+n)\prod\limits_{j=1}^{D}(d_i,m)(d_i',n)m!n!} \quad (B.7)$$

where $A + B < C + D + 1$ or $A + B < C + D + 1$ and

$x + y < \min(1,2^{C-D+1})$

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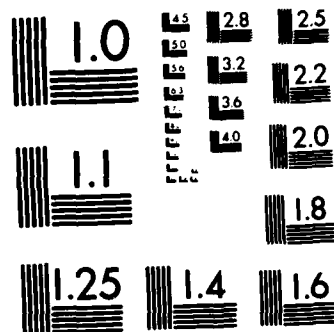
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APPENDIX C

Kellogg-Barnes Distributions

APPENDIX C: Kellogg-Barnes Distributions

Kellogg-Barnes Type I distribution

$$f_{X,Y}(x,y) = \frac{4\alpha^{\beta+1}}{\pi\Gamma(\beta+1)} (x^2 + y^2)^\beta e^{-\alpha(x^2 + y^2)} , \quad \begin{array}{l} x,y>0 \\ \alpha,\beta>0 \end{array}$$

The probability density function, (p.d.f.), of X for $\beta=1$ is given as

$$f_X(x) = \frac{4\alpha^2}{\pi} e^{-\alpha x^2} \int_0^\infty (x^2+y^2)e^{-\alpha y^2} dy$$

$$= \frac{2\alpha^{3/2}}{\sqrt{\pi}} (x^2 + \frac{1}{2\alpha})e^{-\alpha x^2}$$

By symmetry, the same is true for the p.d.f of y.

moments of the distribution

The moments of the distribution may be derived from equations (1.5) - (1.11). By symmetry, it is clear that $\mu_x=\mu_y$ and $\sigma_x^2=\sigma_y^2$.

$$\mu_x = \frac{4\alpha^{\beta+1}}{\pi\Gamma(\beta+1)} \int_0^\infty \int_0^\infty x(x^2 + y^2)^\beta e^{-\alpha(x^2 + y^2)} dxdy$$

Letting $r^2 = x^2+y^2$, $x = r\cos\theta$, and $y = r\sin\theta$ yields

$$\mu_x = \frac{4\alpha^{\beta+1}}{\pi\Gamma(\beta+1)} \int_0^\infty r^{2\beta+2} e^{-\alpha r^2} \int_0^{\pi/2} \cos\theta \, d\theta dr$$

Performing the inner integral and letting $z=r^2$ and $dz=2rdr$ gives

$$\mu_x = \frac{2\alpha^{\beta+1}}{\pi\Gamma(\beta+1)} \int_0^\infty z^{\beta+1/2} e^{-\alpha z} dz$$

$$= \frac{2\Gamma(\beta+3/2)}{\pi\sqrt{\alpha}\Gamma(\beta+1)}$$

The variance may be given by

$$\sigma_x^2 = \int_0^\infty \int_0^\infty x^2 f_{X,Y}(x,y) \, dxdy - \mu_x^2$$

$$= \frac{4\alpha^{\beta+1}}{\pi\Gamma(\beta+1)} \int_0^\infty \int_0^{\pi/2} r^{2\beta+3} e^{-\alpha r^2} \cos^2\theta \, d\theta dr - \mu_x^2$$

$$= \frac{\beta+1}{2\alpha} - \frac{4}{\pi^2\alpha}\left[\frac{\Gamma(\beta+3/2)}{\Gamma(\beta+1)}\right]^2$$

From (1.10), the covariance of X,Y is given by

$$\text{cov}(x,y) = \int_0^\infty \int_0^\infty xy f_{X,Y}(x,y) \, dxdy - \mu_x\mu_y$$

$$= \frac{4\alpha^{\beta+1}}{\pi\Gamma(\beta+1)} \int_0^\infty r^{2\beta+3} e^{-\alpha r^2} \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta dr - \mu_x\mu_y$$

$$= \frac{\beta+1}{\pi\alpha} - \frac{4}{\pi^2\alpha}\left[\frac{\Gamma(\beta+3/2)}{\Gamma(\beta+1)}\right]^2$$

Since the correlation for X,Y is the covariance divided by the standard deviations of X and Y, the correlation $\rho(x,y)$ may be given

by

$$\rho(x,y) = \frac{2}{\pi}\left(\frac{\beta+1-\pi a}{\beta+1-2a}\right)$$

where

$$a = \frac{4}{\pi^2}\left[\frac{\Gamma(\beta+3/2)}{\Gamma(\beta+1)}\right]^2$$

Kellogg-Barnes Type II distribution

$$f_{X,Y}(x,y) = \beta\alpha^2 e^{-(\alpha x + \beta y/x)} , \qquad \begin{array}{l} x,y>0 \\ \alpha,\beta>0 \end{array}$$

The p.d.f. of X is given by

$$f_X(x) = \beta\alpha^2 \int_0^\infty e^{-(\alpha x + \beta y/x)} \, dy$$

$$= \alpha^2 xe^{-\alpha x}$$

Similarly, the p.d.f. of Y is given by

$$f_Y(y) = \beta\alpha^2 \int_0^\infty e^{-(\alpha x + \beta y/x)} \, dy$$

Recognizing that $e^{-\alpha x}$ is the kernel of the Laplace transform, the p.d.f. may be given by

$$f_y(y) = L_\alpha \left\{ \beta\alpha^2 e^{-\beta y/x} \right\}$$

and from (95:146 #25)

$$f_Y(y) = 2(\beta\alpha)^{3/2} \sqrt{y} \; K_1[\; 2(\alpha\beta y)^{1/2} \;]$$

where $K_\nu[z]$ is the modified Bessel function as defined by Erdelyi
(95:371;8:5). The final form for $f_Y(y)$ can also be derived by taking
the Mellin transform of $f_{X,Y}(x,y)$ and then taking the limit as $s \to 1$,
(15:27 #3.16).

moments of the distribution

$$\mu_x = \int_0^\infty \int_0^\infty x\beta\alpha^2 \; e^{-(\alpha x + \beta y/x)} \; dxdy$$

$$= 2/\alpha$$

$$\mu_y = \int_0^\infty \int_0^\infty y\beta\alpha^2 \; e^{-(\alpha x + \beta y/x)} \; dxdy$$

$$= 2/\alpha\beta$$

$$\sigma_x^2 = \int_0^\infty \int_0^\infty x^2 \; \beta\alpha^2 \; e^{-(\alpha x + \beta y/x)} \; dxdy - \mu x^2$$

$$= \alpha^2 \int_0^\infty x^3 \; e^{-\alpha x} \; dx - 4/\alpha^2$$

$$= 2/\alpha^2$$

$$\sigma_y{}^2 = \int\limits_{0}^{\infty}\int\limits_{0}^{\infty} y^2 \, \beta\alpha^2 \, e^{-(\alpha x + \beta y/x)} \, dxdy - \mu_y{}^2$$

$$= \frac{2\alpha^2}{\beta^2} \int\limits_{0}^{\infty} x^3 e^{-\alpha x} \, dx - 4/(\alpha\beta)^2$$

$$= 8/\alpha\beta^2$$

$$\text{cov}(x,y) = \int\limits_{0}^{\infty}\int\limits_{0}^{\infty} xy\beta\alpha^2 \, e^{-(\alpha x + \beta y/x)} \, dxdy - \mu_x\mu_y$$

$$= \frac{\alpha^2}{\beta} \int\limits_{0}^{\infty} x^3 \, e^{-\alpha x} \, dx - 4/\beta\alpha^2$$

$$= 2/\beta\alpha^2$$

$$\rho(x,y) = \frac{\text{cov}(x,y)}{\sigma_x \sigma_y}$$

$$= 1/2$$

Kellogg-Barnes Type III distribution

$$f_{X,Y}(x,y) = \frac{\beta\alpha^c}{\Gamma(c)} \, x^c \, e^{-(\alpha x + \beta xy)} \quad , \qquad \begin{array}{l} x,y>0 \\ \alpha,\beta>0 \ , \ c>2 \end{array}$$

The p.d.f of X is given by

$$f_X(x) = \frac{\beta\alpha^c}{\Gamma(c)} \, x^c \, e^{-\alpha x} \int_0^\infty e^{-\beta xy} \, dy$$

$$= \frac{\alpha^c}{\Gamma(c)} \, x^{c-1} \, e^{-\alpha x}$$

Similarly, the p.d.f. of Y is given by

$$f_Y(y) = \frac{\beta\alpha^c}{\Gamma(c)} \int_0^\infty x^c \, e^{-(\alpha+\beta y)x} \, dx$$

$$= \frac{\beta\alpha^c \, \Gamma(c+1)}{\Gamma(c)(\alpha+\beta y)^{c+1}}$$

$$= c\beta\alpha^c \, (\alpha + \beta y)^{-(c+1)}$$

moments of the distribution

$$\mu_x = \frac{\beta\alpha^c}{\Gamma(c)} \int_0^\infty \int_0^\infty x^{c+1} \, e^{-(\alpha x + \beta xy)} \, dxdy$$

$$= c/\alpha$$

$$\mu_y = \frac{\beta \alpha^c}{\Gamma(c)} \int_0^\infty x^c \, e^{-\alpha x} \int_0^\infty y e^{-\beta xy} \, dy dx$$

$$= \alpha/\beta(c-1)$$

$$\sigma_x^2 = \frac{\beta \alpha^c}{\Gamma(c)} \int_0^\infty x^{c+2} \, e^{-\alpha x} \int_0^\infty e^{-\beta xy} \, dy dx - \mu_x^2$$

$$= \frac{\alpha^c}{\Gamma(c)} \int_0^\infty x^{c+1} \, e^{-\alpha x} \, dx - (c/\alpha)^2$$

$$= c/\alpha^2$$

$$\sigma_y^2 = \frac{\beta \alpha^c}{\Gamma(c)} \int_0^\infty x^c \, e^{-\alpha x} \int_0^\infty y^2 \, e^{-\beta xy} \, dy dx - \mu_y^2$$

$$= \frac{2\alpha^c}{\beta^2 \Gamma(c)} \int_0^\infty x^{c-3} \, e^{-\alpha x} \, dx - \frac{\alpha^2}{\beta^2(c-1)^2}$$

$$= \frac{\alpha^2 c}{\beta^2 (c-1)^2 (c-2)}$$

$$\text{cov}(x,y) = \frac{\beta\alpha^c}{\Gamma(c)} \int_0^\infty x^{c+1} e^{-\alpha x} \int_0^\infty y e^{-\beta xy} \, dy\,dx - \mu_x\mu_y$$

$$= \frac{\alpha^c}{\Gamma(c)} \int_0^\infty x^{c-1} e^{-\alpha x} \, dx - \frac{c}{\alpha} \beta\frac{\alpha}{(c-1)}$$

$$= 1/\beta(1-c)$$

$$\rho(x,y) = \frac{\text{cov}(x,y)}{\sigma_x\sigma_y}$$

$$= -\frac{\sqrt{c-2}}{c}$$

APPENDIX D

Contour Plots of the Bivariate

H-function Distribution

APPENDIX D: <u>Contour</u> <u>Plots</u> <u>of</u> <u>the</u> <u>Bivariate</u> <u>H-function</u> <u>Distribution</u>

The following contour plots were accomplished using the Vector General 3404 graphic display with the VAX 11/780 computer, VAX/VMS operating system 3.5, Advanced Graphics Lab., University of Texas.

<u>McKay's</u> <u>bivariate</u> <u>gamma</u> <u>distribution</u>

$$f_{X,Y}(x,y) = \frac{a^{p+q}}{\Gamma(p)\Gamma(q)} x^{p-1} (y-x)^{q-1} e^{-ay} \qquad \begin{array}{l} y>x>0 \\ a,p,q>0 \end{array}$$

$$= \frac{a^2}{\Gamma(p)} \, {}_1^{} H_{1,1,0,0,0,1}^{1,0,0,0,1,0} \left[\begin{array}{c|c} ax & (p+q-2,1) \\ & (p+q-1,1) \; ; \; \text{-----} \\ ay & \text{-----} \\ & (p-1,1) \; ; \; \text{-----} \end{array} \right]$$

Plots of McKay's bivariate gamma distribution are shown in Figure D.1.
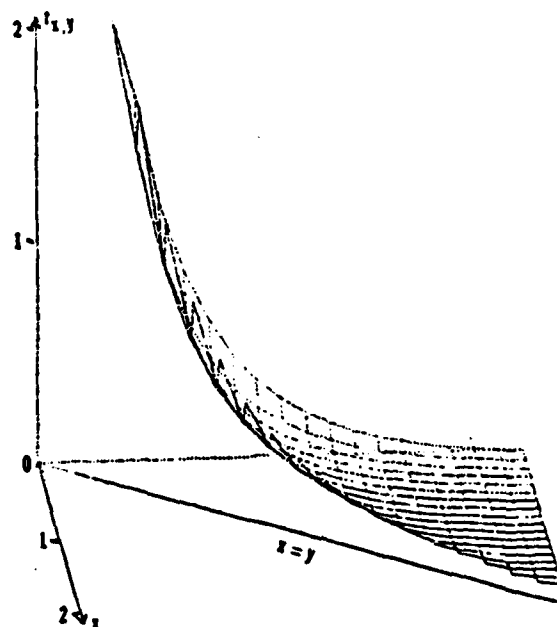
a) a=2.0 , p=q=0.5

Figure D.1 Contour Plot of McKay's Bivariate Gamma Distribution
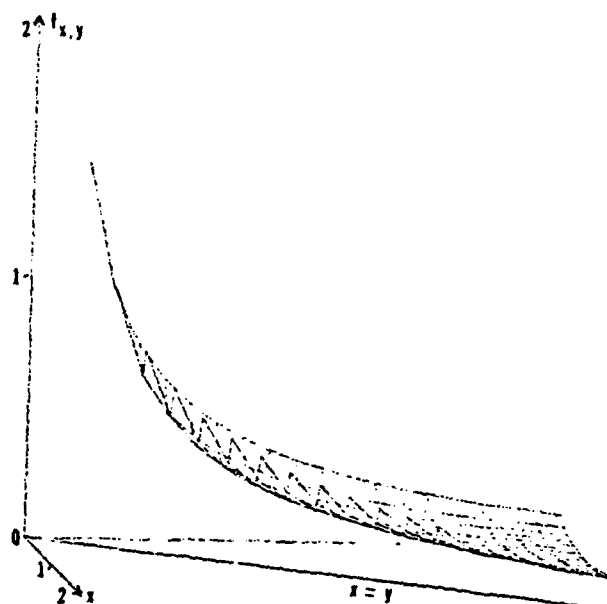
b) a=p=q=0.5
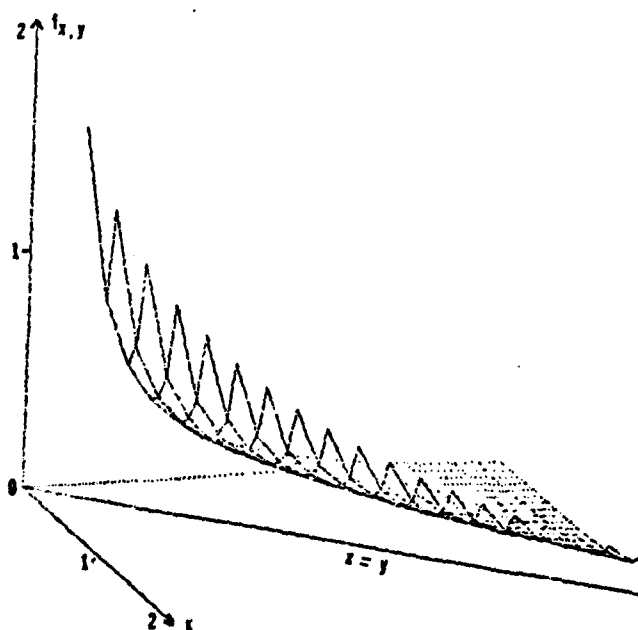
c) a=1.0 , p=0.2 , q=0.8

Figure D.1 Contour Plot of McKay's Bivariate Gamma Distribution
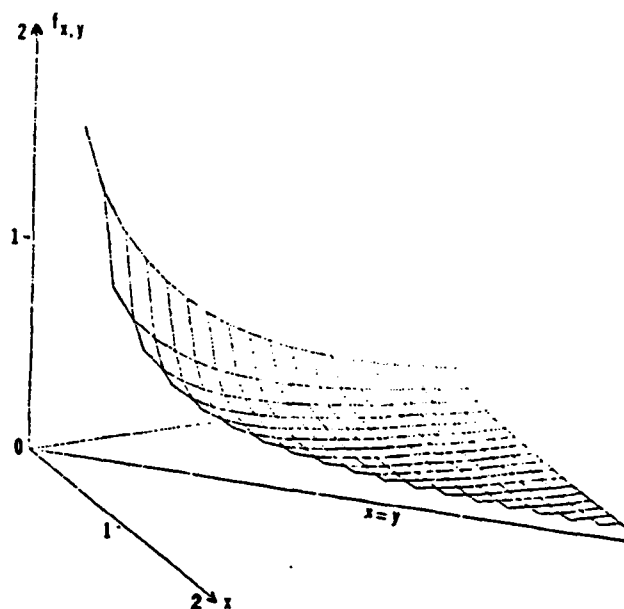
d) a=1.0 , p=0.8 , q=0.2

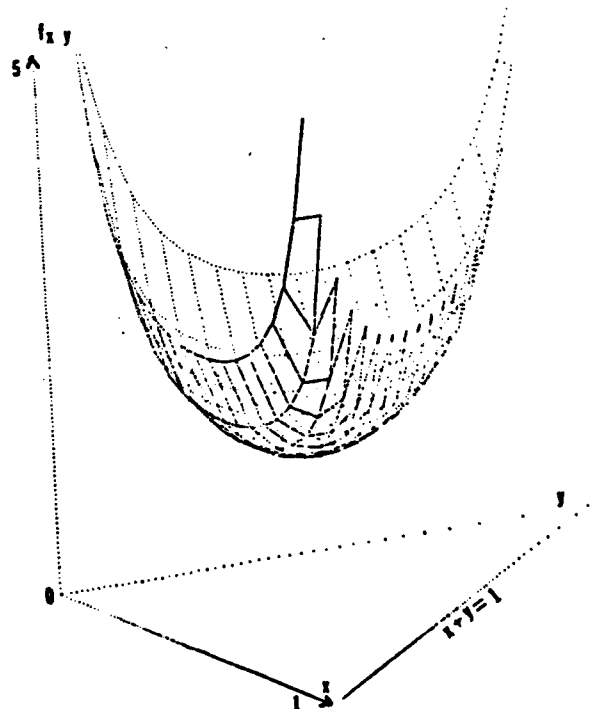Figure D.1 Contour Plot of Mckay's Bivariate Gamma Distribution

The bivariate beta distribution

$$f_{X,Y}(x,y) = \frac{\Gamma(p_1+p_2+p_3)}{\Gamma(p_1)\Gamma(p_2)\Gamma(p_3)} \; x^{p_1-1} \; y^{p_2-1} \; (1-x-y)^{p_3-1}$$
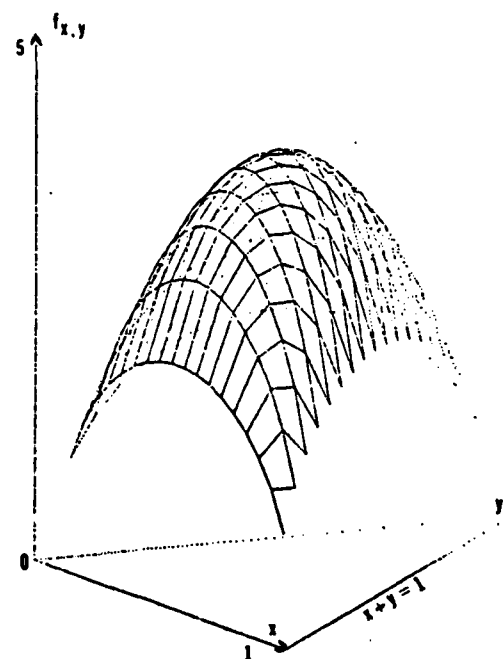
$$x,y>0 \; , \; x+y\leq 1 \; , \; p_1,p_2,p_3>0.$$

$$= \frac{\Gamma(p_1+p_2+p_3)}{\Gamma(p_1)\Gamma(p_2)} \; {}_1H^{1,0,1,0,0,0}_{0,1,0,1,1,0} \left[\begin{array}{c|c} x & \begin{array}{c} \text{-----} \\ \text{-----} \; ; \; \text{-----} \\ (p_1+p_2+p_3-2,1) \\ (p_1-1,1) \; ; \; (p_2-1,1) \end{array} \\ y & \end{array} \right]$$

Plots of the bivariate Beta distribution are shown in Figure D.2.

a) $p_1=p_2=p_3=0.5$

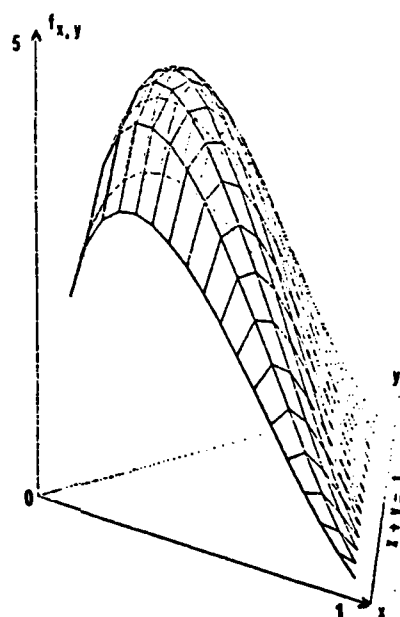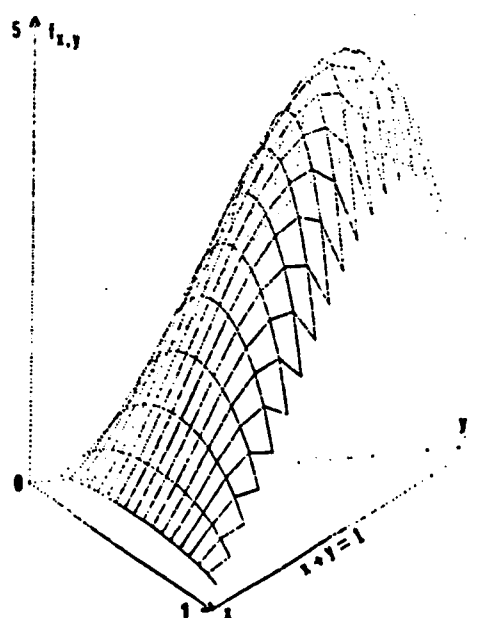Figure D.2 Contour Plot of the Bivariate Beta Distribution

b) $p_1 = p_2 = p_3 = 1.5$
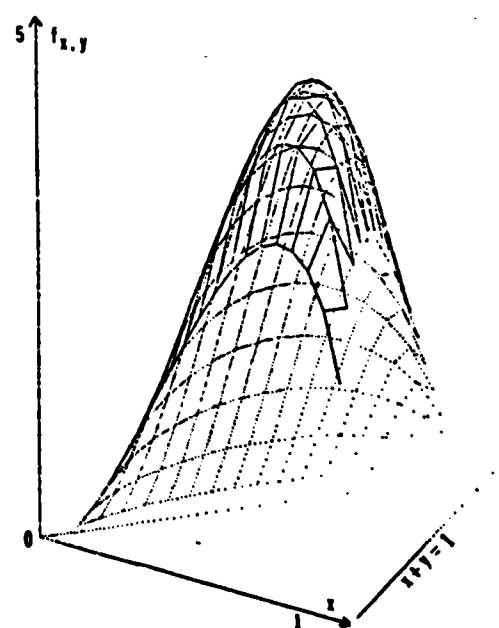
c) $p_1 = 2.5$, $p_2 = p_3 = 1.5$

Figure D.2   Contour Plot of the Bivariate Beta Distribution

d) $p_1 = p_2 = 1.5$, $p_3 = 2.5$
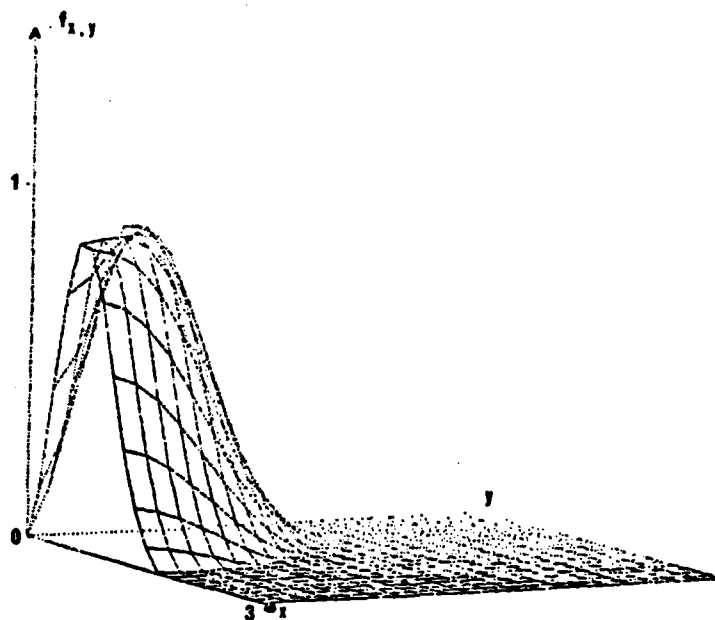

e) $p_1 = p_3 = 1.5$, $p_2 = 2.5$

Figure D.2 Contour Plot of the Bivariate Beta Distribution

Kellogg-Barnes Type I distribution

$$f_{X,Y}(x,y) = \frac{4\alpha^{\beta+1}}{\pi\Gamma(\beta+1)} (x^2 + y^2)^\beta e^{-\alpha(x^2 + y^2)} , \qquad \begin{array}{c} x,y>0 \\ \alpha,\beta>0 \end{array}$$
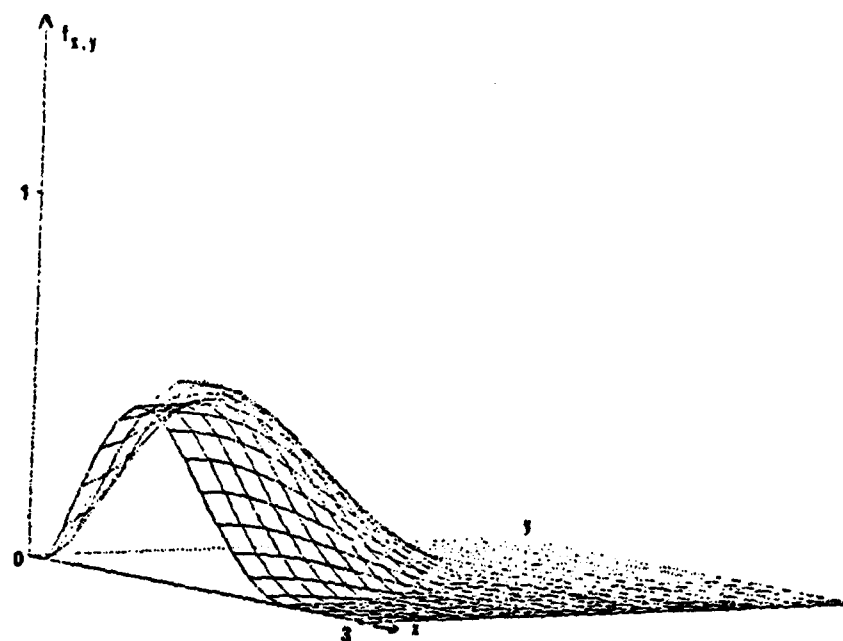
$$= \frac{\alpha}{\pi\Gamma(\beta+1)} \, {}_1H_{0,1,0,1,1,1}^{1,0,1,0,1,0} \left[\begin{array}{c|c} \sqrt{\alpha}\ x & (\beta,1/2) \\ \sqrt{\alpha}\ y & \begin{array}{c} ----- \ ; \ ----- \\ (0,1/2) \end{array} \\ & (0,1/2) \ ; \ (0,1/2) \end{array} \right]$$

Plots of the Kellogg-Barnes I distribution are shown in Figure D.3.
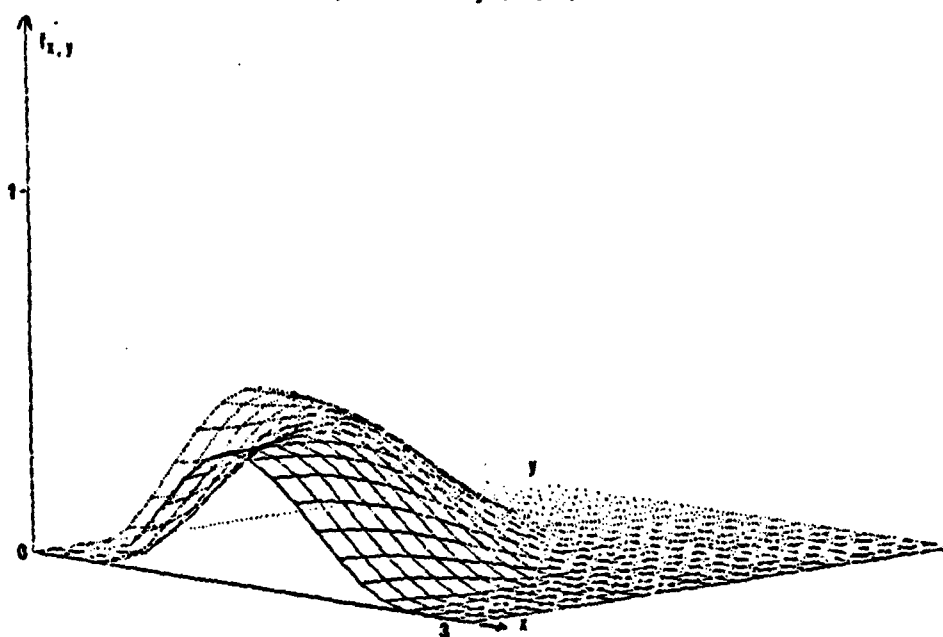

a) $\alpha=2.0$, $\beta=1.0$

Figure D.3 Contour Plot of the Kellogg-Barnes I Distribution

b) $\alpha=1.0$, $\beta=1.0$
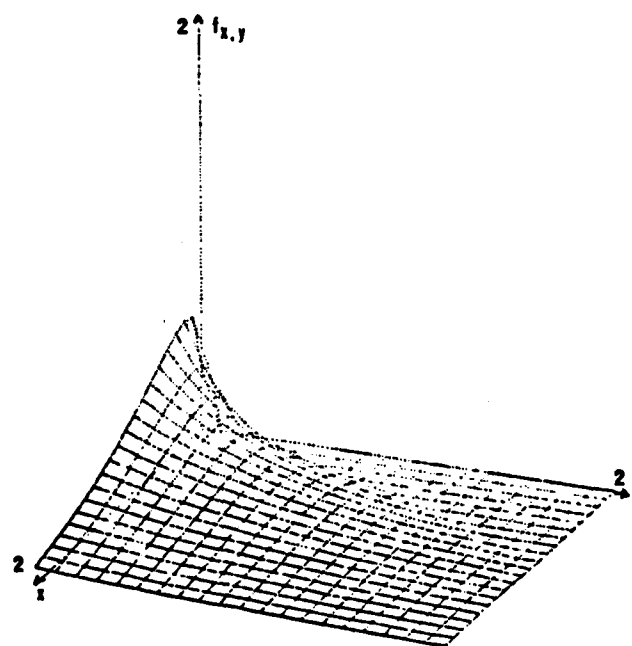

c) $\alpha=1.0$, $\beta=2.0$

Figure D.3 Contour Plot of the Kellogg-Barnes I Distribution

Kellogg-Barnes Type II distribution

$$f_{X,Y}(x,y) = \beta\alpha^2 \, e^{-\alpha x - \beta y/x} \, , \qquad\qquad \begin{array}{l} x,y>0 \\ \alpha,\beta>0 \end{array}$$
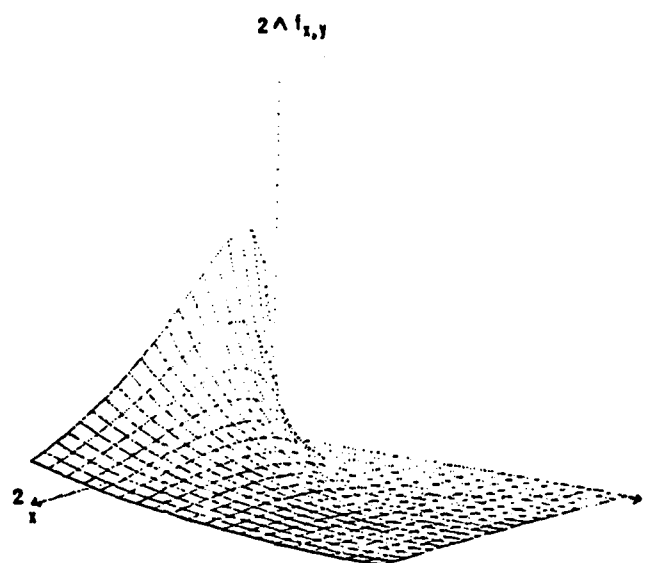
$$= \beta\alpha^2 \; {}_1H \begin{array}{l} 0,0,1,0,1,0 \\ 0,0,0,1,0,1 \end{array} \left[\begin{array}{c|cc} & & (0,1) \\ \alpha x & \text{-----} & ; & \text{-----} \\ \alpha\beta y & & \text{-----} \\ & \text{-----} & ; & (0,1) \end{array} \right]$$

The Kellogg-Barnes II distribution is shown in Figure D.4

a) $\alpha=1.0$, $\beta=1.0$

Figure D.4   Contour Plot of the Kellogg-Barnes II Distribution

$2 \wedge f_{x,y}$

b) α=1.0 , β=2.0

$2 \quad \wedge f_{x,y}$

c) α=2.0 , β=1.0

Figure D.4 Contour Plot of the Kellogg-Barnes II Distribution

Kellogg-Barnes Type III distribution

$$f_{X,Y}(x,y) = \frac{\beta \alpha^c}{\Gamma(c)} x^c e^{-\alpha x - \beta xy} , \qquad \begin{array}{c} x,y>0 \\ \alpha,\beta>0 , c>2 \end{array}$$

$$= \frac{\beta}{\Gamma(c)} 2^H {}^{0,0,0,1,1,0}_{0,0,1,0,0,1} \left[\begin{array}{c|c} & (c,1) \\ \alpha x & ----- ; (1,1) \\ \frac{\beta}{\alpha} y & ----- \\ & ----- ; ----- \end{array} \right]$$

Plots of the Kellogg-Barnes III distribution are shown in Figure D.5.
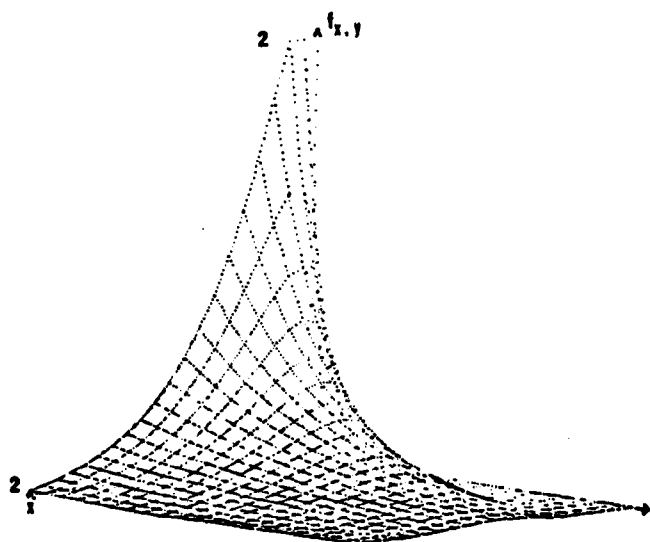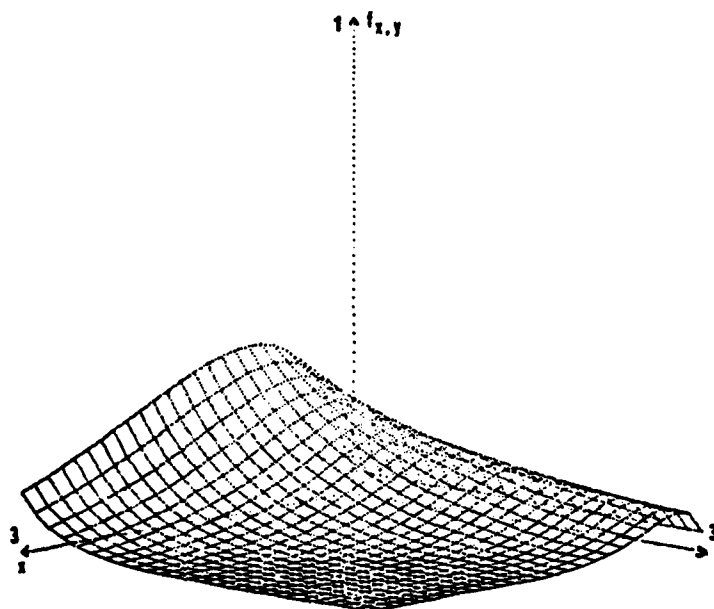

a) $\alpha=\beta=c=1.0$

Figure D.5 Contour Plot of the Kellogg-Barnes III Distribution
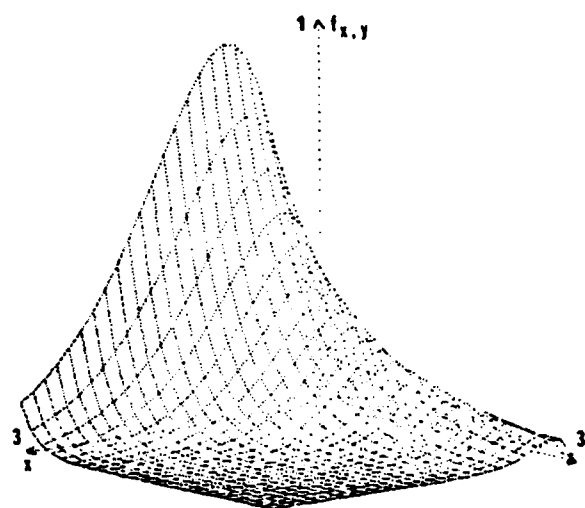
b) $\alpha=\beta=c=2.0$

c) $\alpha=\beta=1.0$, $c=2.0$

Figure D.5 Contour Plot of the Kellogg-Barnes III Distribution

d) α=2.0 , β=c=1.0

e) α=c=1.0 , β=2.0

Figure D.5 Contour Plot of the Kellogg-Barnes III Distribution

f) α=β=2.0 , c=1.0

Figure D.5 Contour Plot of the Kellogg-Barnes Distribution

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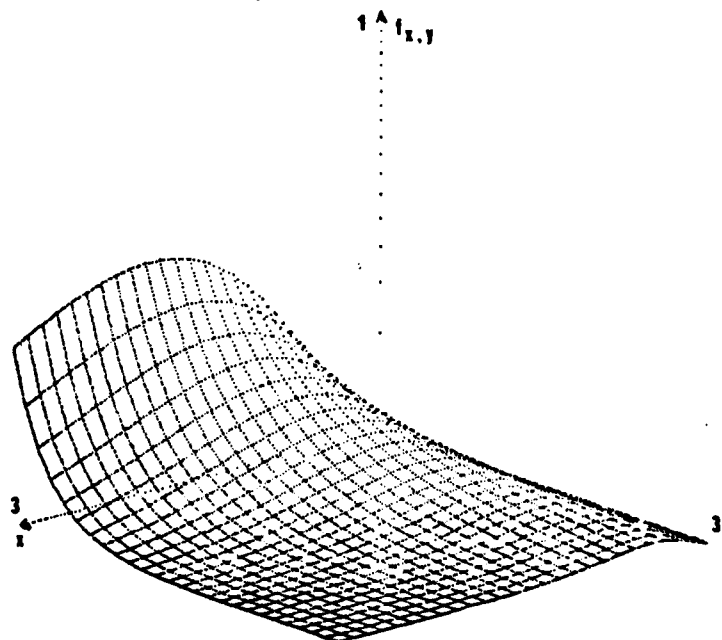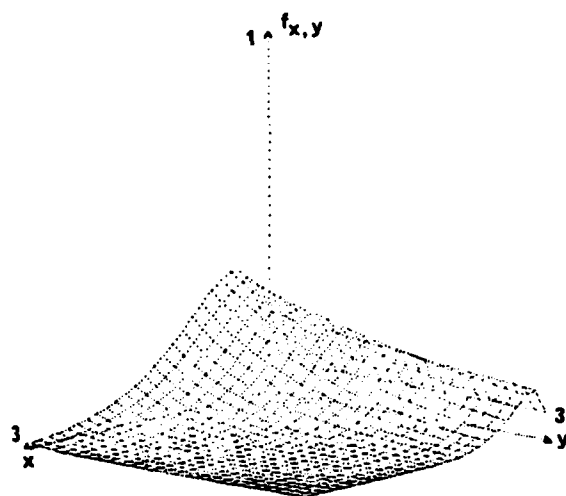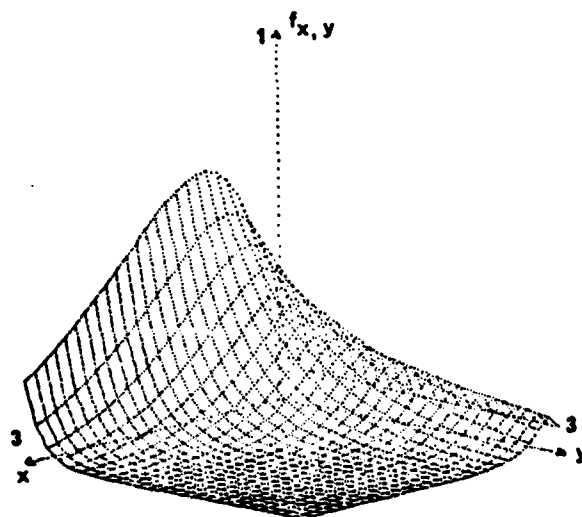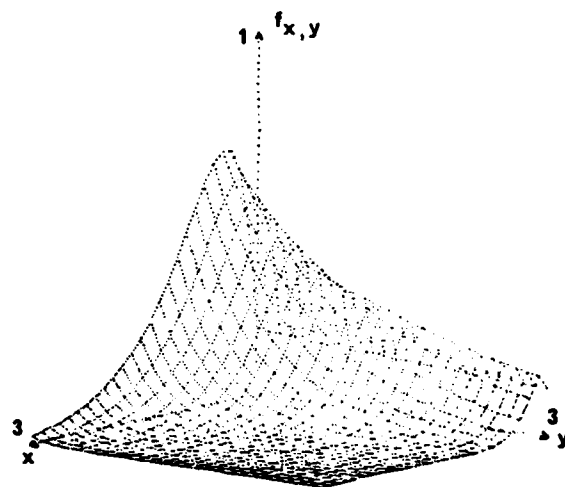A. <u>Primary References</u>

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